## DOCUMENT RESUME

ED 420 495 SE 061 476

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| AUTHOR | Dossey, John A., Ed.; Swafford, Jane O., Ed.; Parmantie, Marilyn, Ed.; Dossey, Anne E., Ed. |
| TITLE | Proceedings of the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (19th, Bloomington-Normal, IL, October 18-21, 199). Volume 2. |
| INSTITUTION | ERIC Clearinghouse for Science, Mathematics, and Environmental Education, Columbus, OH.; International Group for the Psychology of Mathematics Education. North American Chapter. |
| SPONS AGENCY | Office of Educational Research and Improvement (ED), Washington, DC. |
| PUB DATE | 1997-10-00 |
| NOTE | 331p.; For Volume 1, see SE 061 475. |
| CONTRACT | RR93002013 |
| AVAILABLE FROM | ERIC/CSMEE, The Ohio State University, 1929 Kenny Road, Columbus, OH 43210-1080. |
| PUB TYPE | Collected Works - Proceedings (021) -- ERIC Publications (071) |
| EDRS PRICE | MF01/PC14 Plus Postage. |
| DESCRIPTORS | Educational Change; *Educational Technology; Elementary Secondary Education; Higher Education; *Knowledge Base for Teaching; *Mathematics Education; *Research Methodology; *Teacher Attitudes; *Teacher Education; Thinking Skills |

ABSTRACT

The conference proceedings volume for PME-NA-XIX contains a total of 72 reports: 34 research reports; 20 short oral reports; 11 poster session reports; and 7 discussion group reports. Only the research reports are full reports; the others are generally one-page abstracts. The full reports include: (1) "Equity, Teaching Practices, and Reform: Mathematics Teachers Discuss the Impact of the San Jose Mathematics Leadership Project" (Richard S. Kitchen, Joanne Rossi Becker, and Barbara J. Pence); (2) "One Teacher's Solution to Reforming Mathematics Teaching" (Karen Heinz, Margaret Kinzel, Martin A. Simon, and Ron Tzur); (3) "Secondary Mathematics Teachers' Experiences Using a Reform-Oriented Curriculum To Encourage Student Cooperation and Exploration" (Gwendolyn M. Lloyd and Melvin (Skip) Wilson); (4) "A Fourth-Grade Teacher Implements the 'Spirit' of the NCTM Standards" (Diana F. Steele); (5) "Construction and Validation of the Spatial-Symbolic Pattern Instrument" (Donna F. Berlin and Arthur L. White); (6) "Views about Mathematics Survey: Design and Results" (Marilyn P. Carlson); (7) "Changes in Teachers' Beliefs and Assessments of Students' Thinking across the First Year of Implementation of Cognitively Guided Instruction" (Anita H. Bowman, George W. Bright, and Nancy N. Vacc); (8) "Sustaining Cultures of Teaching for Constructive Mathematics Education" (M. Jayne Fleener and Roland G. Pourdavood); (9) "The Educational Benefits of Being a Participant in a Research Study: One Preservice Secondary Mathematics Teacher's Experience" (Deborah A. Gober); (10) "Preservice Mathematics Teachers' Constructions of Gender Equity in the Classroom" (Denise S. Mewborn and Deborah A. Gober); (11) "The Perceptions of Preservice Elementary Teachers about the Integration of Mathematics and Reading" (Kathryn S. Reinke, Kouider Mokhtari, and Elizabeth Willner); (12) "The Impact of Math Apathy Students on One High

School Teacher" (Kenneth L. Shaw and Cylle Rowell); (13) "District-Wide Reflective Teaching in Mathematics: From Changing the Story to Storing the Change" (Christine D. Thomas and Karen A. Schultz); (14) "Teachers' Beliefs about Mathematics as Assessed with Repertory Grid Methodology" (Steven R. Williams, Miriam Pack, and Lena Licon Khisty); (15) "The Geometry Classroom: The Influence of Teachers' Beliefs" (Kay A. Wohlhuter); (16) "Mathematics Students Teachers' Development of Teacher Knowledge and Reflection" (Maria L. Fernandez); (17) "Using Videos To Provide "Case-Like" Experiences in an Elementary Mathematics Methods Course" (Susan N. Friel); (18) "Mathematics Culture Clash: Negotiating New Classroom Norms with Prospective Teachers" (Betsy McNeal and Martin Simon); (19) "A Model for Studying the Relationship between Teachers' Cognitions and Their Instructional Practice in Mathematics" (Alice F. Artzt and Eleanor Armour-Thomas); (20) "Mediating Pedagogical Content Knowledge through Social Interactions: A Prospective Teacher's Emerging Practice" (Maria L. Blanton and Sarah B. Berenson); (21) "Learning To Teach Algebraic Division for Understanding: A Comparison and Contrast between Two Experienced Teachers" (Jose N. Contreras); (22) "Preservice Secondary Mathematics Teachers' Interpretations of Mathematical Proof" (Eric J. Knuth and Rebekah L. Elliott); (23) "Why Do We Invert and Multiply? Elementary Teachers' Struggle To Conceptualize Division of Fractions" (Ron Tzur and Maria Timmerman); (24) "Shape Makers: A Computer Microworld for Promoting Dynamic Imagery in Support of Geometric Reasoning" (Michael T. Battista and Caroline Van Auken Borrow); (25) "Interactive Diagrams: A New Learning Tool" (Jere Confrey, Jose Castro Filho, and Alan Maloney); (26) "Conjecturing and Representational Style in CAS-Assisted Mathematical Problem Solving" (M. Kathleen Heid, Glendon W. Blume, Karen Flanagan, Kenneth Kerr, James Marshall, and Linda Iseri); (27) "Roles of Symbolic Representation in CAS-Assisted Mathematical Problem Solving" (M. Kathleen Heid, Glendon W. Blume, Linda Iseri, Karen Flanagan, Kenneth Kerr, and James Marshall); (28) "Analyzing Students' Learning with Computer-Based Microworlds: Do You See What I See?" (Janet Bowers); (29) "Warning: Asking Questions May Lower Your Mathematical Status in Small Groups" (Kathy M.C. Ivey); (30) "Occasioning Understanding: Understanding Occasioning" (Thomas Kieren, Elaine Simmt, and Joyce Mgembelo); (31) "An Analysis of Students' Development of Reasoning Strategies within the Context of Measurement" (Kay McClain, Paul Cobb, and Koeno Gravemeijer); (32) "Coordinating Social and Psychological Perspectives To Analyze Students' Conceptions of Measurement" (Michelle Stephan and Kay McClain); (33) "Learning as Sense-Making and Property-Noticing" (David Slavit); and (34) "Proportional Reasoning of Early Adolescents: Validation of Karplus, Pulos and Stage's Model" (Linda Gellings, Donald W. Wortham, Abbe H. Herzig, and Dave Eber). (ASK)

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ED 420 495

## Proceedings of the
## Nineteenth Annual Meeting

North American Chapter
of the International Group
for the

# Psychology of Mathematics Education

## Volume 2: Discussion Groups, Research Papers, Short Oral Reports, and Poster Presentations (continued)

October 18-21, 1997
Illinois State University
Bloomington/Normal, Illinois U.S.A.

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Proceedings of the
Nineteenth Annual Meeting

North American Chapter
of the International Group
for the

# Psychology of Mathematics Education

**Volume 2: Discussion Groups,
Research Papers, Short Oral Reports,
Poster Presentations (continued)**

PME-NA XIX

October 18-21, 1997
Illinois State University
Bloomington/Normal, Illinois U.S.A.

Editors:
John A. Dossey
Jane O. Swafford
Marilyn Parmantie
Anne E. Dossey

Published by
The ERIC Clearinghouse for Science,
Mathematics, and Environmental Education
Columbus, OH

**Cite as:**

Dossey, J. A., Swafford, J. O., Parmantie, M., and Dossey, A. E. (Eds.). (1997). *Proceedings of the Nineteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

**Document development:**

David L. Haury, ERIC/CSMEE Executive Editor
Linda A. Milbourne, ERIC/CSMEE Copyeditor
David L. Haury & Linda A. Milbourne, Design & Layout

**Accession number: SE 060 729**

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This publication was funded by the Office of Educational Research and Improvement, U.S. Department of Education under contract no. RR93002013. Opinions expressed in this publication do not necessarily reflect the position or policies of OERI or the Department of Education.

## History and Aims of the PME Group

PME came into existence at the Third International Congress on Mathematical Education (ICME 3) held in Karlsruhe, Germany, in 1976. It is affiliated with the International Commission for Mathematical Instruction.

The major goals of the International Group and of the North American Chapter (PME-NA) are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education;
2. To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics teachers;
3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

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# Preface

This program began with a meeting of interested volunteers in October 1996 at Panama City, Florida during the 18th PME-NA meeting. The results of the ideas discussed and suggestions made were taken to a meeting of the local program committee at Illinois State University where the theme of the psychological underpinnings of mathematics education was selected. This theme became the focus of two of the plenary sessions. Perspectives from cognitive psychology about the foundations of learning mathematics are presented in a paper by James Greeno. Children's intuitions about numbers are discussed from the perspective of developmental psychology in a paper by Robbie Case. A special memorial lecture in honor of Alba Thompson given by Suzanne Wilson is also planned, but the paper dealing with reform and issues surrounding it was not available at the time these proceedings went to press. Alba was both an active member of PME and a former faculty member at Illinois State University. Hence, the local program committee decided to include this memorial to her in the program of the PME meeting held in her former city of residence.

Included in the *Proceedings* are 68 research reports, 9 discussion groups, 40 oral reports, and 41 poster presentation entries. The research reports and the one-page synopses of discussion groups, oral reports, and poster presentations are organized by topics following the pattern begun with the *Proceedings* of the 1994 PME-NA meeting. Additionally, an alphabetical index by author is provided in both volumes. Initially 238 proposals were received with 218 for research reports, 11 oral reports, and 9 discussion groups. Proposals for all categories were blind reviewed by three reviewers with expertise in the topic of submission. Cases of disagreement among reviewers were refereed by a subcommittee of the Program Committee at Illinois State University. The process resulted in the acceptance without reassignment of about 33% of the research report proposals with an overall acceptance rate across all categories of about 58%.

Submissions for the *Proceedings* were made on disk and read by the editors. The format of the papers was adjusted to make them uniform but substantive editing was not undertaken. Papers are grouped by topic area for the table of contents and cross referenced alphabetically in the index to both volumes by the first author.

The editors wish to express thanks to all those who submitted proposals, the reviewers, the 1997 Program Committee, and the PME-NA Steering Committee for making the program an excellent contribution to the growing body of research and discussions about psychology and mathematics education. The Program Chairs would like to extend our special appreciation to the mathematics education faculty at Illinois State University for their support and generous contributions to the preparations for the conference.

Jane O. Swafford
John A. Dossey
October 1997

# Contents

## Contents of Volume 1

History and Aims of the PME Group iii
Reviewers iv
PME-NA Committees v
PME Proceedings in the ERIC Database vi
Preface vii

### Plenary Sessions

Participation as Fundamental in Learning Mathematics 1
*James G. Greeno*

### Advanced Mathematical Thinking

#### Research Reports

An Undergraduate Student's Understanding and Use of Mathematical Definitions in Real Analysis 17
*Barbara S. Edwards*
Mis-Generalization in Calculus: Searching for the Origins 23
*David E. Meel*
Students' Cognitive Approaches to the Concept of Rate 31
*Rodolfo Oliveros and Manuel Santos-Trigo*
Effects of Different Instructional Approaches on Calculus Students' Understanding of the Relationship Between Slope, Rate of Change, and the First Derivative 37
*Donald T. Porzio*
The Process of Periodicity 45
*Gilli Shama and Nitsa Movshovitz-Hadar*
The Relationship Between Written and Verbal Performances: A Study of First Year Calculus Students' Understanding of the Derivative 51
*Kathleen G. Snook*

#### Short Orals

Fostering Students' Construction of Knowledge During a Semester Course in Abstract Algebra 57
*Thomas G. Edwards And Lawrence Brenton*

The Role of General Logic Actions Development in Teaching Mathematics 58
*Mikhail M. Bouniaev*
Learning Rate of Change 59
*Garnet S. Hauger*
Teachers' Connections Among Various Representations of Slope 61
*Sheryl L. Stump*

**Poster Sessions**

Synthesis Between Visual and Analytic Acts in Linear Differential Equations Through a Graphical Argument 62
*Miguel Solis Esquinca & Francisco Cordero Osorio*
On the Relationship Between Conceptual and Algorithmic Aspects in Integral Calculus: An Example in Kinematics 63
*Germán Muñoz Ortega*
Some Representations and Their Interactions With Mental Constructions in the Concepts of Calculus 65
*Francisco Cordero*
Understanding of Relations Between Function and Its Derivative 66
*Francisco Cordero, Daniel Hernández & Asuman Oktaç*
Understanding of Chain Rule in a Graphic Context 67
*Francisco Cordero, Raul Cuellar & Asuman Oktaç*
Methodology to Study Teacher Change in a Reform Calculus Curriculum 68
*Thomas Fox*
The Effect of Context on Students' Procedural Thinking 69
*Ted Hodgson*

## Algebra and Algebraic Thinking

**Research Reports**

Mathematical Patterns in the Middle Grades: Symbolic Representations and Solution Strategies 73
*Joyce Wolfer Bishop*
The Relationship of Undergraduates' Beliefs About Learning Algebra and Their Choice of Reasoning Strategies for Solving Algebra Problems 81
*Albert D. Otto, Cheryl A. Lubinski, & Carol T. Benson*
Teachers' Beliefs and Student Failure in Algebra 91
*Daniel K. Siebert*

### Short Orals

Algebra I Students' Ideas About Covariation 98
*Wendy N. Coulombe & Sarah B. Berenson*
College Algebra: Student Self-Portraits 99
*Victor V. Cifarelli & Tracy Goodson-Espy*
The Meaning of Algebraic Variables: An Empirical Study With Students Aged 16-18 100
*Verónica Hoyos Aguilar*
Content Analysis of Algebraic Word Problems for Gender-Based Reasoning Patterns 102
*Janet M. Herrelko*

### Poster Sessions

Generalization of Patterns and Relationships by Prospective Elementary Teachers 103
*Jane O. Swafford, Cynthia W. Langrall, Gladis Kersaint*
A Hypertext Resource for the Teaching and Learning of School Algebra 104
*Daniel I. Chazan*

## Assessment

### Research Reports

Mandated Assessment Instruments: How Do Teachers Value Them? 107
*Karen Bell & Thomas J. Cooney*
Using Assessment Practices as a Tool for Changing Teaching Methodology 115
*Daniel J. Brahier*
A Proposed Method for Assessing Teachers' Pedagogical Content Knowledge 123
*Janet Warfield*
Assessing Student Work: The Teacher Knowledge Demands of Open-Ended Tasks 131
*Linda Dager Wilson & Patricia Ann Kenney*

### Short Orals

Assessment as Dialogue in College Developmental Mathematics: Gains in Student and Instructor Understanding 138
*Richard P. Connelly & Cynthia M. Smith*

Assessing Mathematical Knowledge With Concept Maps and Interpretive Essays 139
*Linda A. Bolte*

**Poster Session**

Different Strategies Used to Solve Constructed Response SAT Math Items 140
*Delwyn L. Harnisch*

## Curriculum Reform

### Research Reports

Mathematical Activities in Insurance Agents' Work 143
*Judit Moschkovich*
A Semiotic Framework for Linking Cultural Practice and Classroom Mathematics 151
*Norma C. Presmeg*
Educating Non-College Bound Students: What We Can Learn From Manufacturing Work 157
*John P. Smith III*

### Short Oral

Changes of Presentation in Middle School Mathematics Textbooks: The Case of Integer Operations 163
*Yeping Li, Jack Carter & Beverly Ferrucci*

## Data, Probability, & Statistics

### Research Reports

Probability Instruction Informed by Children's Thinking 167
*Graham A. Jones, Carol A. Thornton & Cynthia W. Langrall*
Student Understanding of Statistics: Developing the Concept of Distribution 175
*Melissa Mellissinos, Janet E. Ford & Douglas B. McLeod*

### Short Oral

Pre-Service Secondary School Mathematics Teachers' Understanding of Data Analysis and the Fitted Curve 181
*James A. Marshall*

### Poster Sessions

The Complexity of Teachers' Knowledge: An Experienced Teacher's Struggles When Constructing Pictorial Representations About Conditional Probability 182
*José N. Contreras*

## Functions and Graphs

### Research Reports

A Snapshot of Developmental Algebra Students' Concept Images of Function 185
*Phil DeMarois*
Preservice Teachers' Cognitive Approaches to Variables and Functions 193
*David B. Klanderman*

### Short Oral

Investigating Students' Knowledge and Development of Functions in a Technology-Enhanced Precalculus Class: A Conceptual Framework 199
*Armando M. Martínez-Cruz*

### Poster Sessions

Understanding of The Asymptotic Behavior of a Function 200
*Antonio Arellano, Francisco Cordero & Asuman Oktaç*
Teaching Functions in the Elementary Years 202
*Mindy Kalchman*
Students' Understanding of Functions 203
*Lisa Clement*
Assessment of Students' Ability to Represent a Scientific Experiment by an Algebraic Model 204
*John Layman & Gilli Shama*

## Geometry and Geometric Thinking

### Research Reports

The Development of Students' Notions of Proof in High School Classes Using Dynamic Geometry Software 207
*Enrique Galindo With Gudmundur Birgisson, Jean-Marc Cenet, Norm Krumpe & Mike Lutz*

Understanding Angle Ideas by Connecting In-School and
Out-Of-School Mathematics Practice 215
*Joanna O. Masingila & Rapti De Silva*
Defining an Exterior Angle of Certain Concave Quadrilaterals:
The Role of "Supposed Others" in Making a Mathematical
Definition 223
*Yoshinori Shimizu*

**Poster Sessions**

Identifying the Nature of Mathematical Intuitions 230
*Noriyuki Inoue*
When Does the Point Exist in the Plane? Some High School
Students' Conception 231
*Claudia Acuña*

## Instruction/Organization for Instruction

**Research Reports**

Problem-Centered Learning and Early Childhood Mathematics 235
*Noel Geoghegan, Anne Reynolds & Eileen Lillard*
Similarities and Differences of Experienced and Novice K–6
Teachers After an Intervention: The Use of Students' Thinking
in the Teaching of Mathematics 242
*Cheryl A. Lubinski, Albert D. Otto, Beverly S. Rich, & Rosanna Siongco*

**Short Orals**

Fraction Division Interpretations 250
*Rose Sinicrope & Harold W. Mick*
Teachers' Use of Mathematics Curriculum Guides 251
*Janine T. Remillard*

**Poster Sessions**

Collaborative Inquiry for Instructional Change 252
*Ann R. Crawford & Karen Wetherill*
A Review of Reciprocal Teaching and Its Application in Solving
Math Word Problems 254
*Joan J. Marge*
Cooperative Learning in Mathematics: The Effect of Prior
Experience 255
*Rapti De Silva*

## International Studies

### Research Reports

Learning and Teaching Grade 5 Mathematics in New York City, USA, and St. Petersburg, Russia: A Descriptive Study 259
*Frances R. Curcio & Natalia L. Stefanova*
Some Results in the International Comparison of Pupils' Mathematical Views 267
*Erkki Pehkonen*

### Short Oral

Views of German Mathematics Teachers on Mathematics 275
*Günter Törner*

## Language, Discourse & Socio-Cultural Issues

### Research Reports

Teacher Change: Developing an Understanding of Meaningful Mathematical Discourse 285
*Rebekah L. Elliott & Eric J. Knuth*
Group Case Studies of Second Graders Inventing Multidigit Subtraction Methods 291
*Karen C. Fuson & Birch Burghardt*
Teaching Mathematical Procedures Mindfully: Exploring the Conditional Presentation of Information in Mathematics 299
*Ron Ritchhart & Ellen Langer*

### Short Orals

Those Who Talk, Those Who Listen, Ever the Twain Shall Meet: Further Examining The Role of Discourse in the Professional Development of Mathematics Teachers 306
*Dominic D. Peressini And Eric J. Knuth*
The Benefits of Tutoring Programs for Underachieving Students 307
*Yolanda De La Cruz*
Teaching Mathematical Problem Solving to Language Minority Students 309
*Rochelle G. Kaplan*
Student Interactions and Mathematics Discourse: A Study of the Development of Discussions in a Fifth Grade Classroom 311
*Sharon M. Soucy Mccrone*

17

### Poster Sessions

Latino Urban Students and the Meaning of Reform in Mathematics Education 312
*Erick Smith, Michelle Parker & Erin Roche*
Collaborative Problem-Solving in Mixed-Language Groups 313
*Laurie D. Edwards*
Mathematics Attribution Differences by Ethnicity and Socio-Economic Status 314
*Edward Mooney*
Build Your Dream Home: An Ethnomathematical Approach to Motivate the Learning of Mathematics 315
*Jose D. Fonseca*

### Discussion Groups

Discussion Group 4: Applying Socio-Cultural Theories to Research in Mathematics Education 316
*Yolanda De La Cruz, Karen Fuson, Lena Licon-Khisty, Joanna Masingila, Judit Moschkovich, & Betsy Brenner*

Discussion Group 7: Language and Cultural Issues in the Mathematics Classroom 318
*Yolanda De La Cruz, Lena Licón Khisty & Ana Lo Cicero*

## Problem Solving

### Research Reports

Generating Multiple Solutions to Mathematical Problems by Prospective Secondary Teachers 323
*Jinfa Cai*
A Problem Solving Session Designed to Explore the Efficacy of Similes of Learning and Teaching Mathematics 331
*Vilma Mesa & Patricio Herbst*
An Expert's Approach to Mathematical Problem-Solving Instruction 339
*Manuel Santos-Trigo*
Relevance Judgements in Mathematical Problem Solving 347
*Graeme Shirley And Martin Cooper*

### Poster Session

Aspects of Word-Problem Context That Influence Children's Problem-Solving Performance 354
*Lynda R. Wiest*

# Contents of Volume II

## Reform

### Research Reports

Equity, Teaching Practices, and Reform: Mathematics Teachers Discuss the Impact of the San Jose Mathematics Leadership Project 357
*Richard S. Kitchen, Joanne Rossi Becker & Barbara J. Pence*
One Teacher's Solution to Reforming Mathematics Teaching 365
*Karen Heinz, Margaret Kinzel, Martin A. Simon & Ron Tzur*
Secondary Mathematics Teachers' Experiences Using a Reform-Oriented Curriculum to Encourage Student Cooperation and Exploration 371
*Gwendolyn M. Lloyd & Melvin (Skip) Wilson*
A Fourth-Grade Teacher Implements the "Spirit" of the NCTM *Standards* 377
*Diana F. Steele*

## Research Methods and Instuments

### Research Reports

Construction and Validation of the Spatial-Symbolic Pattern Instrument 387
*Donna F. Berlin & Arthur L. White*
Views About Mathematics Survey: Design and Results 395
*Marilyn P. Carlson*

### Short Oral

College Students' Self-Motivation and Mathematical Belief Structure 404
*Martin Risnes*

### Discussion Group

Discussion Group 6: Caricatures Versus Case Studies: A Discussion of Two Methods of Research Reporting 405
*Ronald V. Preston & Diana V. Lambdin*

## Teacher Beliefs

### Research Reports

Changes in Teachers' Beliefs and Assessments of Students' Thinking Across the First Year of Implementation of Cognitively Guided Instruction 409
*Anita H. Bowman, George W. Bright, And Nancy N. Vacc*

Sustaining Cultures of Teaching for Constructive Mathematics Education 417
*M. Jayne Fleener & Roland G. Pourdavood*

The Educational Benefits of Being a Participant in a Research Study: One Preservice Secondary Mathematics Teacher's Experience 423
*Deborah A. Gober*

Preservice Mathematics Teachers' Constructions of Gender Equity in the Classroom 429
*Denise S. Mewborn & Deborah A. Gober*

The Perceptions of Preservice Elementary Teachers About the Integration of Mathematics and Reading 435
*Kathryn S. Reinke, Kouider Mokhtari & Elizabeth Willner*

The Impact of Math Apathy Students on One High School Teacher 441
*Kenneth L. Shaw & Cylle Rowell*

District-Wide Reflective Teaching in Mathematics; From Changing the Story to Storing the Change 447
*Christine D. Thomas & Karen A. Schultz*

Teachers' Beliefs About Mathematics as Assessed With Repertory Grid Methodology 453
*Steven R. Williams, Miriam Pack & Lena Licon Khisty*

The Geometry Classroom: The Influence of Teachers' Beliefs 459
*Kay A. Wohlhuter*

### Short Orals

Change in Teaching Practice During an Action Research Collaborative 464
*Thomas G. Edwards And Sarah Hensien*

Pre-Service K-12 Teachers of Mathematics: Change in Beliefs 465
*David R. Erickson & Georgia A. Cobbs*

### Poster Sessions

Mathematics Proficiency as Viewed Through a Novice Teacher's Goals, Beliefs, and Orientations 466
*Bridget Arvold*

Developing Self Confidence in Mathematics: an Experiment With Preservice Elementary School Teachers 467
*Sunday A. Ajose*

### Discussion Group

Discussion Group 8: The Influence of Beliefs on Behavior and Thus on Learning 469
*Anita N. Kitchens & Jeannie Hollar*

## Teacher Education

### Research Reports

Mathematics Student Teachers' Development of Teacher Knowledge and Reflection 473
*Maria L. Fernández*

Using Videos to Provide "Case-Like" Experiences in an Elementary Mathematics Methods Course 479
*Susan N. Friel*

Mathematics Culture Clash:Negotiating New Classroom Norms With Prospective Teachers 487
*Betsy McNeal & Martin Simon*

### Short Orals

A Growth of Knowledge of a Middle School Teacher 494
*Pi-Jen Lin, Thomas Post, Kathleen Cramer, Marlene Pinzka, Susan Haller, & Joel Burgeson*

Cognitive and Affective Predictors of Preservice Teacher Effectiveness in Elementary Mathematics Pedagogy 496
*Oakley D. Hadfield*

Goals for Understanding Mathematics in Content Courses for Prospective Elementary School Teachers 498
*Virginia L. Keen*

Examining Elementary Teacher Beliefs About Mathematics, Mathematics Teaching, and Technology 499
*Melissa B. Hanzsek-Brill*

Awareness of the Standards, Mathematics Beliefs, and Classroom Practices of Mathematics Faculty at the Collegiate Level 500
*Victoria Boller Laberge & Alan Zollman*
Increasing Prospective Secondary School Mathematics Teachers' Understanding of the Complexity of Teaching 501
*Laura R. Van Zoest*
Pre-Service Teachers' Validations of Mathematical Solutions 502
*Sharon B. Walen & Dianne S. Anderson*
Using History to Further the Understanding of Mathematical Concepts 504
*Libby Krussel*

**Poster Session**

Replicating Combinatorics Research : The Experience of Preservice Teacher "Researchers" 506
*Janet Shroyer, Michelle Mckinley & Norma Shunta*

**Discussion Groups**

Discussion Group 2: Challenges of Reform in Teacher Preparation: Establishing an Agenda for Documenting Meaningful Change Through Research 508
*David R. Erickson, Georgia A. Cobbs & Libby Krussel*
Discussion Group 3: Practicing Teachers Becoming Teacher Educators: What Issues Arise as Teachers Conduct Seminars for Their Peers? 509
*Virginia Bastable & Deborah Schifter*
Discussion Group 5: Principles for Middle Grades Teacher Enhancement:a Participatory Session for Teacher Educators 510
*Kathleen Cramer, William Bush, Richard Lesh, Thomas R. Post, Sid Rachlin, Barbara Reys, & Robert Reys*
Discussion Group 9: Undergraduate Mathematics for Elementary Teachers 512
*Jean Mcgehee, David Molina & Marilyn Carlson*

## Teacher Knowledge

**Research Reports**

A Model for Studying the Relationship Between Teachers' Cognitions and Their Instructional Practice in Mathematics 517
*Alice F. Artzt & Eleanor Armour-Thomas*

Mediating Pedagogical Content Knowledge Through Social Interactions: a Prospective Teacher's Emerging Practice 527
*Maria L. Blanton & Sarah B. Berenson*
Learning to Teach Algebraic Division for Understanding: A Comparison and Contrast Between Two Experienced Teachers 535
*José N. Contreras*
Preservice Secondary Mathematics Teachers' Interpretations of Mathematical Proof 545
*Eric J. Knuth & Rebekah L. Elliott*
Why Do We Invert and Multiply? Elementary Teachers' Struggle to Conceptualize Division of Fractions 553
*Ron Tzur & Maria Timmerman*

**Short Orals**

Understanding Mathematics Curriculum: Teachers Making Sense of Problem Solving, Communication, Connections, and Reasoning 560
*Florence Glanfield*
The Influence of One Teacher's Knowledge of her Students' Mathematical Thinking on her Practice 561
*David Feikes*
Procedural and Conceptual Knowledge in One Teacher's Explanations 563
*José N. Contreras*
Experienced Secondary Teachers' Knowledge of Division and Its Relationship to Multiplication: Two Case Studies 565
*José N. Contreras*

**Poster Session**

A Comparison of Student Beliefs: High School Pre-Algebra and College,Developmental Math 567
*Jeannie Hollar & Anita Kitchens*

## Technology

**Research Reports**

*Shape Makers*: A Computer Microworld for Promoting Dynamic Imagery in Support of Geometric Reasoning 571
*Michael T. Battista & Caroline Van Auken Borrow*
Interactive Diagrams: A New Learning Tool 579
*Jere Confrey, Jose Castro Filho, & Alan Maloney*

Conjecturing and Representational Style in CAS-Assisted Mathematical Problem Solving 585
*M. Kathleen Heid, Glendon W. Blume, Karen Flanagan, Kenneth Kerr, James Marshall & Linda Iseri*
Roles of Symbolic Representation in CAS-Assisted Mathematical Problem Solving 593
*M. Kathleen Heid, Glendon W. Blume, Linda Iseri, Karen Flanagan, Kenneth Kerr, & James Marshall*

**Short Orals**

The Role of Technology in Lesson Planning: The Case of Preservice Teachers 601
*Draga Vidakovic, Sarah Berenson & Maria Blanton*
Teacher as Bricoleur: Using Computing Technologies for Teaching Mathematics 602
*Lynn Stallings*

**Poster Sessions**

Graphing Calculators in the Mathematics Classroom 603
*Laurie J. Riggs*
Cabri-Geometry: A Catalog of Problems That It Solves and Questions Which It Proposes 604
*Eugenio Díazbarriga*
Generating Meanings Beyond "Right-Wrong" Evaluations in an Interactive Manipulative Environment 606
*Sergei Abramovich & Gwendolyn Stephens*

**Discussion Group**

Discussion Group 1: "Psychology" of the TI-92 607
*Donald Porzio & Roger Day*

## Student Knowledge

**Research Reports**

Analyzing Students' Learning With Computer-Based Microworlds: Do You See What I See? 611
*Janet Bowers*
Warning: Asking Questions May Lower Your Mathematical Status in Small Groups 619
*Kathy M. C. Ivey*

Occasioning Understanding: Understanding Occasioning 627
*Thomas Kieren, Elaine Simmt & Joyce Mgembelo*

An Analysis of Students' Development of Reasoning Strategies Within the Context of Measurement 635
*Kay McClain, Paul Cobb & Koeno Gravemeijer*

Coordinating Social and Psychological Perspectives to Analyze Students' Conceptions of Measurement 643
*Michelle Stephan & Kay McClain*

Learning as Sense-Making and Property-Noticing 649
*David Slavit*

Proportional Reasoning of Early Adolescents: Validation of Karplus, Pulos and Stage's Model 657
*Linda Gellings, Donald W. Wortham, Abbe H. Herzig & Dave Eber*

**Short Orals**

A Vygotskian Model for Unfolding, Formulating, and Solving Math Problems From Children's Narratives of Experience 664
*Ana María Lo Cicero*

Different Approaches to the Semiotic Systems Of Representation: Representation In A Mathematical Activity 666
*Fernando Hitt-Espinosa*

Jill's Use of Deductive Reasoning: A Case Study From Grade 10 667
*David A Reid*

**Poster Sessions**

The Development of Children's Understanding of Additive Commutativity 668
*Arthur J. Baroody, Sirpa Tiilikainen, & Jesse L. M. Wilkins*

The Effects of Problem Size on Children's Understanding of Additive Commutativity 669
*Sirpa Tiilikainen, Arthur J. Baroody & Jesse L. M. Wilkins*

Kindergartners' Understanding of Additive Commutativity Within the Context of Word Problems 670
*Jesse L. M. Wilkins, Arthur J. Baroody, & Sirpa Tiilikainen*

Children's Representations of Multiplication 671
*Denise Rowell, Karen S. Norwood, & Maria Blanton*

**Index of Authors** 675

# REFORM

# EQUITY, TEACHING PRACTICES, AND REFORM: MATHEMATICS TEACHERS DISCUSS THE IMPACT OF THE SAN JOSE MATHEMATICS LEADERSHIP PROJECT

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This paper reports on an interview with four high school mathematics teachers involved for three years in an intensive staff development project. We identified changes in beliefs and concomitant classroom practices, as well as requisites and deterrents to change.

## Introduction

During the 1993-1994 school year, two of the authors commenced an extensive staff development project with high school mathematics teachers in the South San Francisco Bay area. Three broad goals guided the National Science Foundation (NSF) funded San José Mathematics Leadership Project[1] (henceforth referred to as "the project"): 1) To facilitate the on-going professional development of mathematics teachers as recommended by the National Council of Teachers of Mathematics (NCTM, 1991); 2) To enhance the content knowledge of mathematics teachers; and 3) To update and enhance the pedagogical knowledge of mathematics teachers to enable them to meet the challenges of proposed curricular innovations (see Becker & Pence, 1996).

The NCTM reform documents (1989, 1991, 1995) have called for changes in K-12 mathematics curriculum, instruction, and assessment. In addition, governmental agencies (see The Task Force on Women, Minorities, and the Handicapped in Science and Technology, 1988) and business and industry leaders in the United States (see Johnston & Packer, 1987) have advocated such reforms. In response to these calls for change the project was begun as a multi-year extension to the one-year Equity 2000 staff development project, a nationwide program initiated in six cities in 1991.

In this paper, we report our findings from an interview conducted with four teachers who had been involved with the project from its inception. The main interest of the authors was to ascertain whether the project had impacted teachers' beliefs and classroom practices. Various researchers have highlighted the

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[1] The project discussed in this paper was supported by grant #TPE 9155282 from the National Science Foundation Teacher Enhancement Program, 1993-1997. The opinions expressed in this paper are those of the authors and do not necessarily reflect those of the National Science Foundation.

27

importance of teacher beliefs to change teacher behavior (Cooney & Jones, 1988; Ernest, 1991). Though connections between teachers' beliefs about mathematics and their classroom practices have been made (Ernest, 1991; Carpenter & Fennema, 1992), beliefs are also influenced by other factors in the context of the school and classroom (Peluso, Becker, Pence, 1996). In this study, we asked the teachers a series of questions to learn about school- and classroom-level factors that influenced their beliefs and practices.

## **Participants**

Four European American high school mathematics teachers, two women and two men, were interviewed in the Spring of 1996. All four interviewees had participated for three years in the project. Ms. Gray had 18 years of experience teaching high school mathematics at various schools. Ms. Ruth had taught for 10 years at a large urban high school in which the composite minority population were in the majority. Mr. Davis had 34 years of teaching experience while Mr. Sims had spent 28 years teaching mathematics at the high school level. Both worked in the same suburban school district, though at different schools. The teachers' schools ranged in size from 1200 to 4000 students, with a representation of students of color ranging from 20% to 75%.

## Methodology

The approach to voice scholarship chosen in this study was narrative inquiry. This research methodology allowed the research participants to define the important issues in their own terms (Riessman, 1993). In the spirit of a narrative inquiry 2approach to qualitative research, we asked the research participants seven broad questions to motivate discussion, supplemented by four probe questions to illuminate various aspects of the respondents' responses. While our principal goal was to learn about how the teachers' involvement in the project had impacted their beliefs and practices, other issues came to the fore during conversations that ensued.

## Research Findings

We classified the themes that emerged from the interview with the four, third-year project participants into the following four categories: the teachers' classroom practices, issues related to equity, requisites identified for classroom reform, and deterrents to change.

### Classroom Teaching Practices

Throughout the interview, the teachers discussed their practices as aligned with the "reform movement" as opposed to more traditional mathematics instruction. To highlight this difference, sample narratives that the teachers provided are presented to illustrate how their practices differed from their "traditional" colleagues.

Mr. Sims described one of his typical lessons:

> A typical class for me to answer your question is ten minutes of explanation of what the activity, no not even ten, five minutes perhaps at the maximum at the beginning to talk about, to set the overview for today's lesson. The students move into their teams and then interact with teams, and I have team rules that are on the front board that have been established since September. Essentially, it (my class) is very student-centered. We use computers on roughly 40% of the time because we have a wonderful computer lab that we got from an Apple grant.

Ms. Gray explained that in her classes she requires her students to write:

> My students are doing portfolios, especially in Algebra II . . . Some (say) . . . that they did more writing in math (in her classes) than they do in English as they formulate their ideas.

Mr. Davis reflected upon his practices in the context of his use of the *College Preparatory Mathematics: Major Change from Within* (CPM) textbook series (Sallee & Kysh, 1990):

> *Three years ago, four years ago, I was doing the usual routine; We had a textbook and you would cover, either cover the new material first and then answer questions on the old material or vice versa. It was pretty teacher-oriented, and very rarely did they (the students) have any time in class to work on anything. With CPM, it's very difficult to do anything but student-centered because the textbook doesn't have any examples.*

Though these quotes provide only small insights into the teachers' practices, they give a sense of how the teachers viewed their teaching as distinct from many of their colleagues'. Among these mathematics teachers who considered themselves "reform" teachers, large differences were present in their beliefs and corresponding practices. Certainly, many discrepancies also exist among teachers who may classify themselves as "traditional." Putting these variations among like-minded teachers aside, the critical point is that all four of the participants in this interview considered themselves adherents to the reform agenda in mathematics education. This ascribed allegiance differentiated them from many of their colleagues who they frequently referred to as "traditional" teachers.

**Equity Issues**

A variety of issues related to equity emerged during the interview session. The teachers defended the reform agenda as a means to more appropriately teach mathematics to the majority of students. Here there was evidence that the project had substantially changed teachers' expectations for students which translated into changes in curriculum and instruction. Mr. Sims stated:

> When she (a speaker) said all kids take Algebra One/Geometry/Algebra Two in that booming voice of hers . . . and then she described the population . . . these are not sons and daughters of Stanford professors. . . . [That] made a huge impact on myself and two others in our group. We never would be where we are because we thought we were doing the right things for our kids [by tracking].

In Ms. Ruth's district, all 9th graders were enrolled in Algebra as part of the Equity 2000 program. In light of this fact, Ms. Ruth argued that it was inappropriate to teach many of her students college preparatory mathematics, demonstrating her less than full acceptance of Equity 2000 goals. However, she pointed out that in order for students to be successful, instruction must change:

> But we cannot teach math in the same traditional way. If you have Algebra for All students coming at the 9th grade, what kind of algebra can you teach these kids?... I'm sorry to say, we are still basically following the traditional Algebra-Geometry-Algebra type of curriculum and our failure rate is extremely high because of that.

Ms. Gray asserted that reforms in mathematics education were positively impacting the majority of students who were not college bound:

> *You see kids who are blossoming also and changing direction paths. I gave this young lady who is struggling to get a D the problem about car racing. . . . She did research into newspapers and it was a door opening (experience) for her, an encouragement.*

An overriding goal of the teachers was that all of their students have access to a powerful mathematics curriculum. At the time of the interview, approximately 70% of the students at Mr. Sims' school, who in the past would have been placed in the lower-tracked Math A/Math B sequence, were passing Algebra One. In reference to these students' grades in Algebra One, Mr. Sims said:

> *Now we're not talking about As and Bs, we're talking about Cs and Ds. But still we look at this and said, 'What would have happened to these kids otherwise?'*

Throughout these conversations, all of the teachers raised issues pertinent to equity. In particular, the teachers believed that all students need to study algebra.

**Requisites for Classroom Change**

Aside from the obvious requirement to learn about the reform, the teachers identified a number of important necessities to assist their change efforts, including: networking, a critical mass of support within school, and availability of reformed textbooks.

Ms. Ruth and Mr. Sims agreed that the greatest benefit of the project was having the opportunity to meet other people. Mr. Sims stated:

> *I think that one of the things that has been the most valuable for me is to meet the teachers from other schools and other school districts and to share ideas with them and find out what is going on at the different schools.*

This networking, while not an explicit goal of the project, emerged as a recurrent theme in this and other research with project participants (Becker & Pence, 1996).

Discussions that occurred throughout the interview generally reiterated the importance of having a majority or a critical mass of the mathematics faculty committed to change for the reform effort to be successful at the school-level. Because five out of the eight mathematics teachers on Mr. Sims' faculty had been actively involved in the project, "tremendous change has happened at my school as a result of this (project)."

The acquisition of innovative textbooks played an important role in changing the mathematics curriculum at several schools. While explaining their typical lessons, both Mr. Davis and Mr. Sims discussed the importance of the text to create a student-centered classroom. Furthermore, certain textbooks allowed for the incorporation of technology more than others. Mr. Sims discussed the benefits of the direct inclusion of technology in textbooks in the context of his work load:

> *In Geometry, I love Sketchpad. It works for me again because I have a program in place - Serra [1989]. Serra has the program Sketchpad directly tied to the course. You don't have to stay up until midnight to figure out what you are going to do the next day. It's there, it's ready, and it's wonderful.*

An issue that also emerged was dependence on the text to implement reform. Given the teachers' heavy workload, perhaps it is not surprising that these teachers wanted reformed curriculum materials to facilitate their efforts to change how they taught mathematics.

## Deterrents to Change

The teachers identified two major obstacles that they faced in their efforts to change how they taught mathematics: lack of peer support and the insufficient alignment of assessment to reform. Mr. Davis who had 34 years of teaching experience said:

> *A lot of people have been around . . . and they saw, I've seen what happened to SMSG (it failed). They're saying maybe this is just another fad like that SMSG . . . and so we'll wait and see if it catches on.*

A barrier to reform that cannot be overlooked is the lack of support from, and even the resistance of, one's peers. Ms. Ruth expressed her frustration:

> I know that at Chavez High School when I suggested something it was like talking to a brick wall. . . . They don't want to change.

The teachers also discussed their classroom assessment practices, as well as standardized, large-scale examinations. Ms. Gray would have liked to have learned more about assessment through her participation in the project:

> *This curriculum change has taken us three years to do and to change your assessment takes research, it takes other people to support you, it takes too much. I think that's part of the project that needs, that didn't get touched enough. We changed the curriculum, but no one has changed how they grade.*

Mr. Sims characterized how assessment should also reflect his efforts to engage his students in mathematical discovery. Furthermore, changing one's assessment practices was critical to implement reform:

> *I agree with Ms. Gray, assessment has to drive the curriculum, especially if you want to have change and we haven't done enough with it. I want to change but again it's a time issue. Gee, do I have time to really sit down and think about what I'd like to have done in the classroom? We are already (working) 10-12 hour days and to change into something as important as this, you've got to have something ready.*

The teachers were concerned about the failure of large-scale examinations such as the SAT to reflect the goals of the reform movement. Ms. Gray indicated that this lack of alignment made her nervous. "Yeah," said Mr. Sims, "we have to care about our kids!" These two experienced, caring mathematics teachers were obviously conflicted between teaching mathematics in a less traditional manner and preparing their students for high-stakes, large-scale examinations.

## Conclusion

The teachers' narratives demonstrated that their involvement in the project had impacted their beliefs about issues related to equity. The teachers discussed the importance of access to algebra for all students and the concurrent need to change instruction. The value of networking, the value of a "critical mass" of support from colleagues from the same school, and the availability of reformed textbooks were identified by the teachers as requirements for classroom change. In contrast, the teachers described how a lack of peer support, and inadequate alignment of their classroom assessment practices and standardized examinations to reform goals as the primary barriers to change.

## References

Becker, J. R., & Pence, B. J. (1996). Mathematics teacher development: Connections to change in teachers' beliefs and practices. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education.* Valencia, Spain: University of Valencia.

Carpenter, T. P., & Fennema, E. (1992). Cognitively guided instruction: Building on the knowledge of students and teachers. *International Journal of Educational Research, 17* (5), 457-470.

Cooney, T., & Jones, D. (1988). *The relevance of teachers' beliefs for research in mathematics teacher education.* Paper presented at the Sixth International Congress on Mathematics Education, Budapest, Hungary.

Ernest, P. (1991). The impact of beliefs on the teaching of mathematics. In J. A. Dossey, A. E. Dossey, & M. Parmantie (Eds.), *Preservice teacher education: The papers of action group 6 from ICME 6, Budapest, Hungary* (237-242). Normal, IL: Illinois State University.

Johnston, W. B., & Packer, A. E. (Eds.). (1987). *Workforce 2000: Work and workers for the twenty-first century.* Indianapolis, IN: Hudson Institute.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1995). *Assessment standards for school mathematics.* Reston, VA: Author.

Peluso, T., Becker, J. R., & Pence, B. J. (1996). Extending the educational conversation: Site based staff development. in E. Jakubowski, D. Watkins, & H. Biske (Eds.). *Proceedings of the Eighteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Riessman, C. K. (1993). *Narrative analysis.* Newbury Park, CA: Sage Publications.

Sallee, T., & Kysh, J. (1990). *College preparatory mathematics: Major change from within.* Davis, CA: Authors.

Serra, M. (1989). *Discovering geometry: An inductive approach.* Berkeley: Key Curriculum Press.

The Task Force on Women, Minorities, and the Handicapped in Science and Technology. (1988). *Changing America: The new face of science and engineering. Interim report.* Washington, DC: Author.

# ONE TEACHER'S SOLUTION TO REFORMING MATHEMATICS TEACHING

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This is an account of a sixth-grade teacher's (Ivy) practice as she responds to the challenges of reforming the content and process of her mathematics teaching. We analyzed two sets of classroom observations and interviews to understand how Ivy teaches new mathematics material to her students. Ivy creates a trajectory for student participation consisting of steps of mathematical knowing and activities to move students from one step to the next. Ivy monitors students' progress along this unchanging trajectory. The understanding of Ivy's practice provided by the account can contribute to understanding the nature of teachers' solutions to the problems posed by the reform movement and the processes by which mathematics teacher development can occur.

## Background

The current mathematics reform agenda (cf. National Council of Teachers of Mathematics, 1989, 1991, 1995) places significant demands on teachers to change the content and process of their mathematics teaching. Teachers respond to this challenge in a variety of ways that reflect their knowledge, beliefs, motivations, and the meanings that they give to their teaching and professional development experiences. Through investigating teachers' participation in the reform, researchers can construct knowledge of teacher development from traditional towards reform practices. This paper presents a theoretical account of the teaching practice that Ivy has developed as she participates in mathematics education reform. By using "practice" we refer to the whole of what the teacher does in the classroom which includes planning, assessing, and interacting with students, as well as everything she knows, believes, and thinks about what she does.

Ivy is a participant in the Mathematics Teacher Development Project (MTD), a 4.5-year research project studying elementary mathematics teacher development. The MTD Project brings together practicing and prospective teachers in an intensive 3-year instructional program designed to promote the development of both groups. Our research includes case studies of practicing teachers. The analysis discussed in this paper is based on data collected prior to the beginning of the MTD instructional program, and reflects Ivy's adaptation to her experiences teaching mathematics and to her prior professional development opportunities, both situated in communities committed to mathematics education reform.

## Theoretical Framework

The MTD Project uses a teacher development experiment methodology (Simon, in press) that combines whole class teaching experiments (Cobb, in press) in teacher education classes with case studies of teachers. The teacher development interventions of the project contribute to and are dependent upon understanding teachers' thoughts, feelings, beliefs, motivations, and actions related to their teaching practice. We organize our understandings of teachers' practices by generating theoretical accounts of that practice. These accounts represent our commitment to articulate how the teacher organizes her experiential realities with respect to teaching. In other words, we strive to create a coherent story of the teacher's practice by *explaining the teacher's perspective from the researcher's perspective* (Simon & Tzur, 1997).

This last phrase distinguishes our work from both deficit studies of teachers and from work in which the teachers articulate their own perspectives (cf. Schifter, 1996). Thinking about "the teacher's perspective from the researcher's perspective" involves a subtle, but important distinction. The researchers attempt to understand and articulate the teacher's approach to the problems of practice: how and what the teacher perceives and how she makes sense of, thinks about, and responds to the situations as she perceives them. However, the result may be very different from what the teacher would say about her own practice. We structure our accounts of the teacher's practice using particular conceptual lenses, often not shared by the teacher, that define our focus and guide our interpretations.

Particular conceptual lenses that we bring to thinking about mathematics teaching are described by Simon (1995). This work characterizes mathematics teaching as including a process of generating and constantly modifying hypothetical learning trajectories (HLT) for students' learning based on the teacher's current (evolving) understandings, particularly of the mathematical terrain and of the students' mathematics. The paradox that we work within is that generating theoretical accounts of teachers' practice involves setting aside our current view of practice in order to conceptualize the teachers' practice. Yet, our current view of practice structures our perception of the teachers' practice and is thus fundamental to what we notice (pay attention to).

## Methodology

We generated our initial account of Ivy's practice based on two sets of data (videotaped classroom observations and audiotaped interviews) collected prior to the beginning of the instructional program. One set of data consists of two consecutive, related (in terms of subject matter) mathematics lessons and interviews before, between, and after the lessons.

The central question that guided our line-by-line analysis of the data was: "How does Ivy endeavor to teach her students mathematics that is beyond what the students already know?" We watched and listened to tapes with accompanying transcripts. Each section of the data that seemed to pertain directly or indirectly to the central research question was discussed, resulting in either tentative inferences about what it revealed or questions about how those data might be interpreted. As the analyses proceeded, we generated hypotheses to explain larger chunks of data, and revised these hypotheses as indicated by subsequent data.

## Ivy's Practice

Ivy is a sixth-grade teacher in a district that promotes reform principles. The district provides a list of grade-level outcomes pertaining to specific content areas, giving teachers the freedom and responsibility to create learning activities and materials for their students.

Ivy views her experiences as a student in traditional mathematics classes as having failed to foster understanding. In the context of her teacher preparation program and her classroom teaching, Ivy has developed her ability to understand mathematics and has used this ability to improve her understanding of particular mathematics. Based on her experience both as a teacher and learner of mathematics, Ivy believes that students learn better through active participation and "seeing" for themselves. Her preference is to avoid telling students what they should know. As an alternative, Ivy structures her lessons so that her students work in small groups, use manipulatives, and share their ideas during whole class discussions. While analyzing the data, we noted the role of particular sequences of activities in Ivy's lessons and her responses to students' contributions. Following is an articulation of our account of Ivy's practice.

Ivy's goal is to make the mathematics meaningful for her students. For Ivy, particular mathematics is meaningful when one sees how that mathematics has come about. Towards this end, Ivy devises a trajectory intended to build some aspect of mathematics through a set of intermediate steps of knowing. These intermediate steps, or "landmarks", are connected by activities designed to move the students from one landmark to the next. Thus, Ivy's lessons are organized by a trajectory that represents her ideas on how the mathematics comes to have meaning. Students' understanding of the mathematics is a result of their navigating successfully each step in that trajectory.

Ivy's primary commitment is to maintain student progress along the trajectory. Our analyses of the choices she makes throughout her lessons suggest that she uses this trajectory as a lens through which she considers students' contributions. She highlights and supports contributions that have the potential to advance most of the students along her trajectory. In contrast, when students contribute ideas that are deviations from the trajectory, Ivy uses a variety of techniques to direct them back to the trajectory, i.e., towards the next landmark. Below, we present data that illustrate a teaching trajectory devised by Ivy, and how the trajectory influences her responses to students' contributions.

Ivy's goal for this particular lesson was for the students to figure out the formula for the area of a triangle. Ivy created a learning trajectory based on her sense of what makes the formula meaningful, selecting some activities from resource materials. Ivy explained, "We are building off those right triangle ideas because that is where the formula builds from which is actually from rectangles. So I am trying to take them from rectangles to right triangles to non-right triangles."

Her trajectory proceeded through the following steps: (a) Children use the geoboard to discover a way to determine the area of a right triangle. Ivy intended that they would find the area of the circumscribed rectangle and divide it by two. (b) Children compute the area of as many different right triangles as they can produce on the geoboard. (c) Ivy creates a chart to record the base, height, and area of the triangles for which the students determined the area. (d) Ivy presents the children with the problem of finding a pattern, a formula, to compute the area of a triangle given its base and height. (e) Students determine the area of non-right (oblique) triangles to determine if the formula holds for non-right triangles.

For us, a key aspect of Ivy's teaching was reflected in her responses to two students, Dave and Jim. When participating in the activity for step (a) above, many of the students initially found the area of the triangle by estimating how many squares the triangle contained. Ivy responded by challenging the students to search for an exact method. Next, Dave doubled the triangle to form a rectangle and divided the area of the rectangle by two. This time, Ivy responded by paraphrasing Dave and emphasizing aspects of his method that were important with respect to the trajectory. She then asked, "How many people like [Dave's solution]?" This is typical of how Ivy responds to contributions that serve to move the students along her trajectory. In contrast, Ivy's response to Jim illustrates how she handles deviations from her trajectory. Jim announced his disagreement with Dave's solution. Ivy encouraged Jim to explain his objection. Jim attempted to explain his conflicting result which was based on an estimation of partial squares. After telling Jim that he could not be sure that his estimations were accurate, Ivy led Jim step-by-step through Dave's method by asking a series of leading questions to which Jim replied with the anticipated one-word answers. Here we see Ivy concentrating on getting Jim to accept Dave's solution and thus to see it as appropriate for use in the next step of the trajectory (step (b) above).

Later in the lesson, Ivy asked the students to find the area of a non-right (and acute) triangle. She expected them to find its area by dividing the triangle into two right triangles. Some of the students took an approach that Ivy did not anticipate: they circumscribed a rectangle around the non-right triangle. In the interview following the class, Ivy talked about her assessment of the students' understanding: "They haven't seen that there are two right triangles created inside . . . and we have got to find the area of the two right triangles." As the interview continued, Ivy explained what she would do if none of the children saw the two right triangles.

> Ivy: Make . . . some sort of obscure rule like . . . "You have a rubber band and it is only big enough to go around four pegs." . . . usually, that makes it so there is no choice but the choice that I am looking for. I mean I feel that I could eventually tell them too. "Look at this. What just happens if I do this?"

In summary, Ivy listens to her students to assess their progress along the trajectory, not to explore their mathematics. For Ivy, learning is a result of traveling the trajectory. When she perceives that the students are deviating from the trajectory, she shepherds them back. She prefers to have students figure things out for themselves, but if necessary, she would ask leading questions, impose constraints, and tell the students what she wants them to know.

## Discussion

Teachers participating in the mathematics education reform are struggling to create useful alternatives to traditional teaching. This reform is not about implementation of a monolithic model. Theoretical accounts of teachers' practice, such as the one presented here, contribute to understanding teachers' interpretations of the reform, the meanings that they give to their teaching and professional development experience, and the types of practice that they generate.

Our account of Ivy's practice suggests that she has a particular orientation toward teaching and learning. Ivy provides her students with opportunities to use their current knowledge to solve non-routine problems, to share their solutions, and to make connections between different mathematical procedures and concepts. However, Ivy views learning as the result of successful movement through a well-designed trajectory. Her role as a teacher then, is to design the trajectory, listen to students' ideas to determine the students' progress along the trajectory (the nature of their ideas does not influence the trajectory), and redirect students' attention back to the trajectory as needed.

Using our conceptual framework for thinking about Ivy's ongoing professional development, we see the opportunity for her to modify her practice to consider and build on the concepts of her students. However, an intervention that would precipitate and facilitate such a change would have to be comprehensive in nature. Our account suggests that Ivy has a coherent practice, one that is likely to structure how she interprets and assimilates professional development opportunities. For example, she may interpret opportunities for her to learn mathematics in a reform-oriented situation as providing trajectories that her students could follow. Opportunities to focus on students' thinking may be seen as ways to know how students are progressing along an envisioned trajectory. Thus, a successful professional development program would have to afAccounts, such as this account of Ivy's practice, can make an important contribution to understanding teacher development. First, these accounts have the potential to enhance understanding of the role of the teacher's knowledge in her practice and of the processes involved in the growth of interrelated areas of teacher knowledge. Second, accounts that characterize the thinking, feeling, acting teacher are needed in order to understand the impact of particular professional development opportunities on teachers' practice. Third, effective teacher education efforts require useful understandings of teachers' current practice. In other words, to create a hypothetical learning trajectory (Simon, 1995) for teacher education, we need accounts of teachers' practice as well as informed visions of where these teacher education efforts might lead.

## References

Cobb, P. (In press). Conducting teaching experiments in collaboration with teachers. In R. Lesh & E. Kelly (Eds.), *Innovative research designs in mathematics and science education*. Dordrecht, Netherlands: Kluwer.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (1995). *Assessment standards for school mathematics*. Reston, VA: Author.

Schifter, D. (Ed.). (1996). *Windows on the classroom: Teachers write about learning and teaching mathematics*. New York: Teachers College Press.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*, 114-145.

Simon, M. (in press). Research on mathematics teacher development: The teacher development experiment. In R. Lesh & E. Kelly (Eds.), *Innovative research designs in mathematics and science education*. Dordrecht, Netherlands: Kluwer.

Simon, M., & Tzur, R. (1997). Generating theoretical accounts of mathematics teachers' practices. *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education*. Lahti, Finland.

# SECONDARY MATHEMATICS TEACHERS' EXPERIENCES USING A REFORM-ORIENTED CURRICULUM TO ENCOURAGE STUDENT COOPERATION AND EXPLORATION

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This paper describes two high school teachers' interpretations of and classroom experiences with a reform-oriented mathematics curriculum. The focus is on the teachers' conceptions of cooperative explorations of mathematical situations. The results elaborate how the curriculum materials presented a challenging vision of instructional practice for one teacher, and a constraint to the fulfillment of a personal vision for another teacher.

Growing consensus in the mathematics education community embraces the notion that if students are to construct meaningful mathematical understandings, their classroom experiences, and therefore the instruction that enables and supports those experiences, must change. In demanding changes in both the content and activity of the mathematics classroom, reform recommendations challenge a lasting tradition of mathematics curriculum and instruction (Gregg, 1995; Richards, 1991; Romberg & Carpenter, 1986). The impressive durability of traditional teacher-centered, procedure-oriented instruction in the mathematics classroom raises the question of how veteran teachers make sense of and deal with calls for reform. This paper describes the conceptions and experiences of two high school teachers attempting to implement a reform-oriented mathematics curriculum that explicitly supports the goals of the *Standards* (National Council of Teachers of Mathematics, 1989). In particular, we focus on the teachers' beliefs about the meaning and importance of cooperation and exploratory problem-solving. How do teachers' beliefs about these issues relate to their interpretations of innovative curricula?

## Participants, Site, And Design

An interpretive case study design (Stake, 1995) was followed to investigate the conceptions and classroom practices of two veteran high school mathematics teachers. Participating teachers were chosen from a public school district in the Northeast U. S. where the curriculum materials of the Core-Plus Mathematics Project [CPMP] were being field-tested. Each teacher joined the project during his or her first year of CPMP implementation. The CPMP materials encourage and support teachers in organizing the classroom so students can work cooperatively to explore mathematical concepts and problems (Hirsch, Coxford, Fey, & Schoen, 1995).

Primary data sources were teacher interviews and classroom observations conducted between September 1994 and January 1997. Because the teachers were participants in a larger ongoing project, the number of interviews and observations (and the time of those activities) varied for the two teachers discussed in this paper. Over a 3-year period, Mr. Allen participated in more than 20 interviews and 100 observations (57 of which were video taped). Ms. Fay was interviewed four times and observed ten times during year 3 of the study (her first year of CPMP implementation). Individual interviews invited teachers to reflect on recent classroom events, and several group interviews allowed teachers to discuss and compare their experiences. Interviews were audio-recorded and transcribed, fieldnotes were taken during classroom observations, and all observations were audio-recorded and approximately half were video-recorded.

## Results

### The Challenges of Curriculum Implementation: The Case of Mr. Allen

Before he began teaching with the CPMP materials, Mr. Allen expressed enthusiasm about the innovative activities and instructional formats: "It sounded neat to me, the idea that it was hands-on, that it was more group work, that the teacher wasn't going to be the focus anymore." Throughout his first three years implementing the curriculum, Mr. Allen repeatedly communicated his overall satisfaction with the CPMP materials and positively differentiated the activities from those found in traditional textbooks. For instance, he indicated that the CPMP activities require students to engage in more "sense-making" by "attacking a problem, reading it, and struggling with it." He was pleased that the activities center on "realistic situations rather than having a bunch of made-up book type problems." Mr. Allen's described how, for example, "The *kids* come up with the objective or understand a little bit more about what a variable can represent based on some practical situations where things are related." His valuation of the "more investigative" approach of CPMP was evidenced on numerous occasions in the classroom when he suggested ways that students could use their "different" approaches and solutions to yield discussion and further learning in the group setting. Mr. Allen believed the cooperative group work in the CPMP program would enhance students' experiences thinking about the problems and give students "more ownership" of the mathematics under consideration. In contrast, he pointed out how "traditional exercises are something that you can do by yourself. There's not a lot of discussion asked or describing asked. It's more or less just get an answer or do some task and get it

42

done. . . . It's not like you can sit in a group and let one person do this part and one person do the other part." His alteration of several CPMP activities to more effectively "split up the work" among group members further evidences the importance to Mr. Allen of cooperative activities *requiring* explicit collaboration among group members.

Although Mr. Allen repeatedly expressed his appreciation for cooperation and exploration components of the CPMP curriculum, there were numerous occasions when he struggled with enacting these components in his classroom. Mr. Allen reported that while working on CPMP problems, students repeatedly asked him questions about "what they need to do." Mr. Allen suggested, "They want to mull over these questions like they're right and wrong rather than thinking about them and giving a basis as to why they answer a question one way or another." Due to his concerns, Mr. Allen often felt compelled to provide more direction than the curriculum materials recommend: "To keep the flow of the class going I want to get in there and help them a little bit." During Mr. Allen's first year, a typical class session involved him circulating among the groups offering directive hints and instructions. In his second and third years, Mr. Allen aimed to help students acquire "a decent sense of what they're doing" by conducting extensive whole-class discussions. Although Mr. Allen believed that students should discuss and learn important ideas in their small groups, he claimed that teacher-directed instruction (particularly in whole-class formats) was necessary to insure that students considered all of the important material. He commented, "Even though I'm giving them more direction, it's giving the students more incentive to keep going and make the attempt on the problems." Mr. Allen also recognized that his concerns about keeping students "on task" and "moving through the materials" caused him "to be the opposite of what [CPMP is] really trying to emphasize" and he expressed a continued desire to "balance it and not go overboard and start doing every problem for them."

For Mr. Allen, the CPMP materials offered a powerful vision of instructional practice that helped him to frame his classroom goals and struggles. Over his three years of implementation of the materials, he maintained a favorable view of the curriculum's philosophy and repeatedly expressed the hope that he would eventually develop a practice more consistent with the "CPMP way." Providing an interesting and informative contrast to Mr. Allen's case, the following section discusses another teacher's experience implementing the CPMP curriculum.

### The Constraints of Curriculum Implementation: The Case of Ms. Fay

When describing her vision of mathematics teaching and learning, Ms. Fay expressed an interest in "giving a really rich problem and letting the students decide how they want to solve it." Problem exploration would involve extensive collaboration among students as they determine the "strategies and tools that we have to solve these problems." In contrast to teacher-directed instruction in which students "are only going to hear it one way," with cooperative work "they hear it from the kids, they hear it from other groups, and it becomes a classroom of everybody teaching and everybody learning." Ms. Fay wished for the CPMP materials to support her in developing her practice to include more student exploration and cooperation. As Ms. Fay began to implement the CPMP materials, she communicated her recognition of many of the things that had originally attracted her to the curriculum, including the students "working together" on "really great problems" with "interesting real-life applications." She expressed that, in contrast to the traditional curriculum's focus on "drill-and-kill" with decontextualized skills and procedures, the CPMP program has "really rich problems and the kids are working their way through them and doing a good job."

Despite the many positive aspects of CPMP, what she encountered as she implemented the curriculum did not coincide as well with her vision as she had hoped. In particular, she communicated disappointment with the opportunities for student exploration and student collaboration within the CPMP materials. Ms. Fay's concern with the limited opportunities for student exploration related primarily to the structure of the activities: "I think the kids are really led through the problem in Core-Plus. . . . Even though they are really good problems, it's handed to them how to solve it." She believed the structured nature of the typical CPMP problem limits students' ability to develop "creative" strategies, solutions, and explanations, and therefore reduces the quality of the cooperative group work that can be based on them. This concern relates to Ms. Fay's view that cooperative work allows students to articulate and extend their understandings through "give-and-take" among group members who have "different ideas." Because the guided problems often left little room for students to "take responsibility for organizing and solving problems in their own ways," the full potential for cooperation to contribute to student exploration, discussion, and understanding was difficult to achieve. In addition, Ms. Fay felt that the CPMP activities do not always require students to work in ways that she considered to be "collaborative." She indicated, "It is hard to give group work that can be done individually" and suggested, "There is not a lot of work in these books [CPMP] that require a shared interest like a jigsaw puzzle where each person takes a piece and we will work together—Each person can work alone."

Given the comments just presented, it is not surprising that Ms. Fay (like Mr. Allen) expressed some concern and experienced some discomfort implementing the group work aspects of the curriculum. She reported not observing very much "give-and-take and discussing of ideas." During several observations, Ms. Fay attempted to compensate for the lack of cooperation among

44

students. For example, at the end of one lesson she engaged her class in a "dominoes" game (not part of the CPMP curriculum) in which the students could "see that if they work together then they will have a better chance of winning than if we do it individually." Ms. Fay felt constrained in her ability to create and implement open-ended problem situations that students could explore more extensively than the suggested CPMP activities. She sensed that she lacked the mathematical background necessary to develop contextualized problems and relevant projects: "I don't know this material. . . . I am learning right along with them so I feel a little more boxed in by it." In addition, she felt that, as a member of her mathematics department "where everybody tries to be at the same place at the same time," she needed to adhere to a particular timeline and activity sequence with the CPMP materials. Her colleagues desire to maintain a uniform time frame and sequence through the materials was most problematic to Ms. Fay because she greatly valued giving students ample time to explore and "interact with each other—not just taking notes and trying to figure out what the teacher wants." Feeling "rushed" and "behind some of the other teachers" contributed to her decisions at times to skip more in-depth or lengthy activities and projects. Ms. Fay indicated that "if everybody did their own thing then [she] would be teaching probably very differently."

## Discussion

The results of this study illustrate the striking differences in how reform-oriented curriculum materials "look" to teachers with different perspectives, goals, and experiences. For both teachers, the structure of problems and activities was a critical feature in their interpretations of different curricula. However, as they described types of mathematics problems, there was remarkable similarity between the comments Mr. Allen made about the traditional curriculum and the comments Ms. Fay made about the CPMP curriculum. For example, Mr. Allen's comment that students in traditional courses can have the attitude "Just tell me the steps I need to do and I'll do those," resembles something Ms. Fay might say about many of the CPMP activities where she fears students are simply "led through the problem." Similar comparisons can be made regarding both teachers' view that cooperative activities should *require* cooperation among participants. This belief contributed to Mr. Allen's disappointment with the "too individual" problems of the school's traditional curriculum and Ms. Fay's disappointment with those of the CPMP curriculum.

Although Mr. Allen never criticized the cooperative nature of the CPMP problems, he did occasionally adapt the activities to involve more "division of the work so students would all have a part." Mr. Allen's more frequent concern regarding group work, which was also voiced by Ms. Fay to a lesser extent, involved difficulties with student behavior during group activities. Both teachers frequently turned to having students work in pairs (rather than groups of four as recommended by the materials). Although the curriculum was designed with the intention of supporting student collaboration and the teachers were motivated to create student-centered classrooms, doing so was not always possible. The teachers' comments and classroom experiences indicate that the CPMP materials do not always provide the types of mathematical activities that can be used by teachers who are committed to requiring or encourage extensive cooperation among students. Whereas teachers' struggles or failures to implement the curriculum "as designed" are frequently attributed to teachers' misunderstanding of, disbelief in, or lack of appreciation for the philosophies underlying the curriculum's proposed activities, these results suggest that teachers may also struggle at least in part due to characteristics of the curriculum itself. These results illustrate how the dynamic relation between teachers' conceptions and particular curricular features (i.e., the cooperative nature of problems) shapes teachers' experiences implementing novel materials in their classrooms.

The ways the two teachers in this study dealt with the cooperation and exploration themes of the CPMP curriculum provides detailed information about what is involved as teachers make sense of reform recommendations and visions. Ms. Fay and Mr. Allen's experiences illustrate the importance to teachers of certain curricular details (e.g., styles of mathematical problems and activities), the types of concerns about students and learning that teachers have as they use new materials, and ways that curriculum implementation provides a context for teacher learning. As we study other teachers' experiences with the same curriculum, we gain even richer insights into how reform-oriented curricula appear to teachers and what implementation of such materials means to them.

## References

Gregg, J. (1995). Tensions and contradictions of the school mathematics tradition. *Journal for Research in Mathematics Education, 26*, 442-466.

Hirsch, C. R., Coxford, A. F., Fey, J. T., & Schoen, H. L. (1995). The Core-Plus Mathematics Project: Teaching sensible mathematics in sense-making ways. *Mathematics Teacher, 88*, 694-700.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author

Richards, J. (1991). Mathematical discussions. In E. von Glasersfeld (Ed.), *Radical constructivism in mathematics education* (pp. 21-47). Dordrecht, The Netherlands: Kluwer.

Romberg, T. A., & Carpenter, T. P. (1986). Research on teaching and learning mathematics: Two disciplines of scientific inquiry. In M. C. Wittrock (Ed.), *The handbook of research on teaching* (3rd ed., pp. 850-873). New York: Macmillan.

Stake, R. E. (1995). *The art of case study research*. Thousand Oaks, CA: Sage.

# A FOURTH-GRADE TEACHER IMPLEMENTS THE "SPIRIT" OF THE NCTM *STANDARDS*

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What happens when an elementary teacher interprets and implements the "spirit" of the NCTM *Standards*? The teaching and learning advocated in *The Standards* (NCTM, 1989, 1991) is based on a constructivist approach. Reshaping mathematics teaching based on a constructivist view of learning presents a considerable challenge to an elementary teacher. This paper describes one teacher's approach to this theory in practice. The researcher, as a participate observer in this classroom, used ethnographic research methods to collect data during each mathematics class for 4 1/2 months. Data analysis revealed 10 instructional strategies the fourth-grade teacher simultaneously used. While four of these strategies will be presented at the oral research report, one specific teaching strategy during a classroom vignette is detailed in this written paper.

Mathematics teachers, teacher educators, and researchers involved in the current reform movement in mathematics education suggest major changes in the teaching of mathematics. They recommend that students become actively involved in constructing their own knowledge and developing mathematical concepts as they explore, explain, and justify solution strategies to mathematical tasks. Through engaging students in communicating about the processes they use to reach solutions, teachers help them construct powerful mathematical knowledge by teaching in contexts of problem solving and reasoning about mathematics. For many teachers, this "constructivist approach" to mathematics teaching requires a change in their conceptions about mathematics and about what it means to teach and learn mathematics (Steele & Widman, 1997). An important step toward changing mathematics teaching is for teachers to see or to read about alternative approaches to teaching.

## Theoretical Framework: Constructivist Learning Theory

The perspective of this researcher is a view of constructivism that combines both cognitive and sociological components of constructivist learning theory. According to the cognitive component of constructivism, students do not passively receive knowledge but actively construct new knowledge based on prior knowledge (Cobb, Yackel, & Wood, 1992). Piaget (1973) suggested that individuals actively construct knowledge internally through their actions on objects in the world and their reflections on these actions. The sociological component of constructivism suggests that the teacher guides the students in communicating their understandings through discourse and interaction (Cobb, 1994). Vygotsky (1978) stated that individuals construct knowledge in the *zone of proximal development* through social interactions with more knowledgeable persons. During interactions with others, individuals come to understand their thinking and initiate changes in their knowledge.

## Methodology

To find a teacher who was implementing a constructivist approach to mathematics teaching and learning in the classroom, this author began by asking administrators, teachers, and parents in several school districts to suggest elementary mathematics teachers who they believed taught mathematics effectively. For a semester, 20 different elementary classrooms were observed and 20 teachers were interviewed. Finally, a fourth-grade teacher, Mrs. Clark, was selected and asked to complete the *Mathematics Beliefs Scales* questionnaire (Fennema, Carpenter, & Peterson, 1987) in order to assess her beliefs about how children learn mathematics, about how mathematics should be taught, and about the relationship between learning concepts and procedures. Her score on the questionnaire confirmed previous observations that this teacher guided students in constructing their own knowledge of mathematics. This teacher agreed to participate in the research.

After Mrs. Clark was identified, data collection was implemented through an ethnographic research approach of participant observation, conducting interviews, and artifact collection. The researcher observed, videotaped, and audiotaped this teacher's mathematics class for 4 1/2 months. Informal and formal interviews were conducted with Mrs. Clark. Her plan book and resource materials, and the children's work were also examined.

Data analysis followed procedures described by Spradley (1980) as the Developmental Research Sequence. This ethnographic research model of data analysis was a cyclic process of questioning, collecting data, recording data, and analyzing data. During the research, this sequence was continually repeated. More than 1000 pages of data were transcribed. Through observations and analyses of the data, ten instructional strategies were identified by the researcher as being consistent with the reform movement in mathematics teaching and learning. Mrs. Clark was not aware that her teaching practice coincided with constructivist learning theory. In order to understand and to appreciate the *whole* quality of Mrs. Clark's teaching, all the strategies must be integrated. However, in order to analyze Mrs. Clark's teaching more subtlely, the strategies need to be separated. While four of these strategies will be presented at the oral research report, one specific teaching strategy during a classroom vignette is detailed in this written paper.

## An Instructional Strategy: Values Students' Thinking

The following vignette provides insight into how Mrs. Clark creates a learning environment that allows students to gain the confidence to construct mathematical knowledge that is powerful and correct. In the vignette, Mrs. Clark calls students by name to get them involved and keep them involved. She listens closely, thus providing a model for students when interacting with one another. She affirms her students' thinking by remembering what they say and using their dialogue to build the lesson. She shows that she values the students' approaches to problems by recognizing their approaches and speaking about them. Notice in the lesson that Mrs. Clark often repeats the students' answers. Sometimes she repeats them verbatim. Other times she paraphrases their answers. She gives credit to students' ideas as she uses their contributions to connect the different ideas. Note there is no judgment of students' verbalized solutions or strategies. Mrs. Clark explains her reasons for the way she responds to students' contributions:

> Sometimes I paraphrase [an answer] because I want the other students in the class to understand it. But I also want the child to know that I am understanding and really listening to what he or she says. If they don't agree that I've done it correctly, they will respond back. I want the other children to hear the idea and value each other's thinking.

The lesson begins:

Mrs. Clark: Can more than one circle have the same center? Raise your hand if you think that more than one circle can have the same center. It looks like we have about half and half.

Hal: Mrs. Clark, can I ask you one thing?

Mrs. Clark: Yes.

Hal: If it's like a circle that goes around and has a point in the middle—Can every one that goes around have the same middle of the circle? (Hal attempts to demonstrate with his fingers what he means.)

Mrs. Clark: That's an excellent question and I didn't think that anybody was going to ask that. I think what he is basically asking—I wish I had something to show it with—I think what Hal is asking is if he has a circle like this.

The teacher draws a figure (see Figure 1).

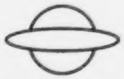

**Figure 1** Drawing of circles in two planes

Mrs. Clark has allowed a student to ask a question of her. She has listened closely to Hal. By validating Hal's thinking, she has helped him believe that his thinking has value. She has valued his thinking so highly that she helps him model and explain his thinking to other students.

But Mrs. Clark does not stop here. She gets two rubber bands and loops one around two fingers and another around two different fingers. She overlaps the rubber bands (see Figure 2), and continues the lesson.

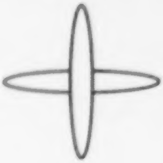

**Figure 2** Overlapping circles using rubber bands

Mrs. Clark: Right, Hal? And the center is in the middle of it. What he is wanting to know is what if I have a circle that goes like this. Is that right?

Hal: Yes.

Hal's idea concerns two circles with the same center that are not in the same plane. Some students disagree with his idea, but Mrs. Clark says she did not say that the circles had to be in the same plane.

Mrs. Clark: So Hal has thought of an example of two [circles] that do have the same center. One circle is headed in this direction in one plane and one in this direction in a different plane.

Hal: I came up with a good one.

Hal shows confidence in his thinking. He has recognized that his thinking is valuable and powerful. According to McLeod (1991), students must believe in their own success in order to lead to greater understanding in their mathematical thinking.

Mrs. Clark: Can anybody think of another way that is possible for more than one circle to have the same center. No one? I saw some diagrams in some people's math logs. Don't be afraid to share your answer. Ann, I saw yours. Come up here and show us your answer.

With this statement, the teacher shows she has been closely observing her students' work. She supports Ann in presenting her idea. Ann draws her representation on the chalkboard (see Figure 3).

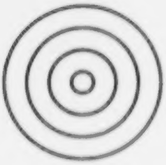

**Figure 3** Ann's drawing

Mrs. Clark: These are called concentric circles. Concentric circles.

The teacher wants others to understand Ann's answer so she asks several students to come up to the front of the classroom. One student becomes the center and others hold hands to form several different circles around the center student. After this demonstration, Missy has a new idea and shows it on the chalkboard (see Figure 4).

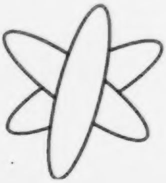

**Figure 4** Missy's drawing

Mrs. Clark: Missy is kind of expanding Hal's idea. She is saying they don't have to be in perpendicular planes, like Hal's. She is saying that I could turn this direction at a slant, move over a little and make another one and so on. Right? Is that still okay?

The teacher has remembered Hal's idea and has shown how it connects to Missy's idea. Mrs. Clark gets out a sphere and puts several rubber bands around it (see Figure 5).

Mrs. Clark: I could have the circle here. (She puts another rubber band around the sphere.) One here. One here. . . . How about that. Isn't that neat? All the ways that I could turn the circle would all have that same. . . .

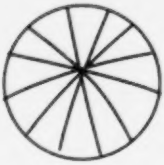

**Figure 5** Connecting Missy and Hal's ideas

Students: Center.

Gerry: There's umpteen times that you could turn that.

Mrs. Clark: What is another word we could use besides umpteen?

Some students: Infinite.

Mrs. Clark: Infinite. Very good. Good job on that. So those of you who said, "no, that circles could not have the same centers," do you now see that there are possible ways that circles could share the same centers?

By listening carefully to her students' ideas and having other students do the same, Mrs. Clark validates the power of their thinking. She seeks to engage *every* student to contribute toward the thinking of the class. She does not just provide general praise of the students' thinking, she names, listens, repeats, paraphrases, and models. She demonstrates that she believes their answers are important, and she values them. As a result, her students eagerly share their answers and are excited about their ideas and their solutions. Several different examples of how more than one circle could have the same center have been presented. Notice that none of the examples came from the teacher. The teacher simply posed the initial problem. The students presented the solutions.

## Conclusions

By taking the students' ideas and thinking seriously, Mrs. Clark has shown students that their ideas and thinking are worthwhile. By allowing them to explore the problem without making judgments, she has shown them it is acceptable to take risks in *this* classroom. The students learned that when they voice their ideas, their ideas will be heard and valued. The students have learned that asking questions is significant in learning mathematics. According to McLeod (1989), if students regularly have positive experiences with mathematics, they will develop attitudes of curiosity and enthusiasm. McLeod noted that two major goals of teaching should be helping students understand the value of their mathematical thinking and helping students develop confidence.

Mrs. Clark created a nonthreatening learning environment in which the students were supported as they communicated their thinking and discovered knowledge for themselves. The teacher verified that each student's thinking is important. She modeled and required students to respect each other's thinking. She affirmed their thinking by repeating their answers and remembering their contributions. Students did not ridicule other students' answers. Students were motivated to learn and showed a positive attitude toward mathematics. They demonstrated confidence in and a willingness to express their thinking.

The "spirit" of the reform movement in mathematics education implies that individuals actively construct knowledge. We need research that shows us what happens when a "real-life" elementary teacher's classroom practice coincides with the reforms in mathematics teaching and learning advocated in *The Standards*. This paper helps contribute to that picture.

## References

Cobb, P. (1994). Where is the mind? Constructivist and sociocultural perspectives on mathematical development. *Educational Researcher, 23*, 13-19.

Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education, 23*, 2-233.

Fennema, E., Carpenter, T. P., & Peterson, P. L. (1987). Mathematics beliefs scales. *Studies of the application of cognitive and instructional science to mathematics instruction.* (National Science Foundation Grant No. MDR-8550236). Madison: University of Wisconsin-Madison.

McLeod, D. B. (1989). The role of affect in mathematical problem solving. In D. B. McLeod & V. M. Adams (Eds.), *Affect and mathematical problem solving* (pp. 20-36). New York: Springer-Verlag.

McLeod, D. B. (1991). Research on learning and instruction in mathematics: The role of affect. In E. Fennema, T. P. Carpenter, & S. J. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 55-82). Albany: State University of New York.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics.* Reston, VA: Author.

Piaget, J. (1973). *To understand is to invent.* New York: Grossman.

Spradley, J. P. (1980). *Participant observation.* Chicago: Holt, Rinehart, and Winston.

Steele, D. F., & Widman, T. F. (1997). Practitioner's research: A study in changing preservice teachers' conceptions about mathematics and mathematics teaching and learning. *School Science and Mathematics, 4*, 184-191.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes.* Cambridge, MA: Harvard University Press.

# RESEARCH METHODS AND INSTRUMENTS

# CONSTRUCTION AND VALIDATION OF THE SPATIAL-SYMBOLIC PATTERN INSTRUMENT

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The ability to understand, relate, and translate patterns presented in different representational modes plays a prominent role in the development of scientific and mathematical literacy. The Spatial-Symbolic Pattern Instrument was designed to measure student's understanding of patterns, defined qualitatively or quantitatively. A principal components analysis of 186 fourth and fifth graders' responses to a 57- item instrument revealed four factors labeled as Figural Pattern (13 items), Monotonic Numeric Pattern (9 items), Word Pattern (5 items), and Nonmonotonic Numeric Pattern (6 items). Reliabilities computed for each of the four subscales, administered as a pretest and as a posttest after pattern instruction, range from 0.81 to 0.95. Statistical analyses suggest a significant interaction effect of grade by gender for the posttest Figural Pattern subscale and a significant grade level effect for the pretest Figural Pattern subscale. Results indicate that the Spatial-Symbolic Pattern Instrument is a reliable and valid instrument to explore student understanding of patterns as presented through different representational modes.

## Introduction

The purpose of this study was to construct and validate an instrument to measure spatial and symbolic processing of patterns by elementary school students. The ability to understand, relate, and translate patterns presented in different representational modes plays a prominent role in the development of scientific and mathematical literacy. The Spatial-Symbolic Pattern Instrument was designed to measure students' recognition, understanding, and application of patterns.

## Theoretical Framework

Two related constructs, pattern and spatial ability, provide the theoretical underpinnings for the development of the Spatial-Symbolic Pattern Instrument. Each will be described and discussed in the context of mathematics and science education curricula.

Current reform literature in both science and mathematics have recognized "patterns" as an essential and fundamental curriculum theme (i.e., one of the "big ideas"; American Association for the Advancement of Science [AAAS], 1989; Steen, 1990). Science has been defined as the search for patterns or regularities in natural and man-made environments (AAAS, 1989) and mathematics as both the language of science and a science of patterns (National

Research Council, 1990).

> What humans do with the language of mathematics is to describe patterns. Mathematics is an exploratory science that seeks to understand every kind of pattern—patterns that occur in nature, patterns invented by the human mind, and even patterns created by other patterns. To grow mathematically, children must be exposed to a rich variety of patterns appropriate to their own lives through which they can see variety, regularity, and interconnections. (Steen, 1990, p. 8)

The study of patterns, hypothesizing rules, and predicting are essential skills that pervade mathematics and science curricula.

According to McGee (1979), spatial ability is composed of two main factors: spatial visualization and spatial orientation. Spatial visualization involves the ability to manipulate and transform mentally two- and three-dimensional objects. Spatial orientation involves the ability to perceive the elements in a pattern, compare patterns, grasp changing orientation in space, and determine the position of one's body in space. The relationship between spatial ability or spatial sense and imagery has been noted by Battista and Clements (1996), Liedtke (1995), Piaget and Inhelder (1967), von Glasersfeld (1982), and Wheatley (1990). The reconstruction of shapes as visual images is not just a matter of isolating perceptual qualities, but involves an active coordination process. This process combines perceptual and proprioceptive elements into relatively stable patterns (Battista & Clements, 1996). It is the mental manipulation of objects, not the perception or retention of visual images, that defines spatial ability.

Students need manipulative and exploratory activities to develop and improve the spatial skills of visualization and orientation (Bruni & Seidenstein, 1990; Wheatley, 1991). These include experiences that focus on relationships; direction, orientation, and perspective of objects in space; the relative shapes and sizes of figures and objects; and how a change in shape relates to a change in size (NCTM, 1989). Spatial reasoning and visualization are linked to the development of mathematics and science concepts and processes (Small & Morton, 1983; Wheatley, 1991).

## Method

The Spatial-Symbolic Pattern Instrument, a 57-item instrument, was designed to measure understanding of pattern presented in figural, numerical, and textual form. Pattern is operationally defined as an arrangement of elements related by a qualitative rule (e.g., organizational, structural, spatial) or a quantitative rule (e.g., numerical, temporal, ordinal). The total sample consisted of 186 students in a small town rural school. Subjects included students in fourth grade (43 boys, 50 girls) and fifth grade (41 boys, 52 girls). Students were randomly assigned to three treatment groups based on pattern instruction using manipulatives, computer simulations, or a combination of these. Students' scores were the total number of correct responses.

Principal components and factor analytical procedures were used to identify constructs underlying students' responses. The Scree method and the percent of explained variance were used to identify four factors. Items with loadings of 0.40 or greater were grouped to comprise the factors. The items included within each of the four factors were combined to comprise four subscales. Group by gender by grade relationships were computed using multivariate analysis of variance, with the four subscales as the dependent variables.

## Findings

The four factors, resulting from the factor analysis, were labeled as: Figural Pattern (FP), Monotonic Numeric Pattern (MNP), Word Pattern (WP), and Nonmonotonic Numeric Pattern (NMNP). The Figural Pattern factor presents the student with incomplete patterns consisting of shapes. The student circles the correct choice to indicate "What comes next?" The Monotonic Numeric Pattern factor presents the student with an incomplete number pattern in ascending or descending order. The student fills in the boxes with the missing numbers. The Word Pattern factor presents the student with a random arrangement of three words representing concepts which the student must order by size. The student writes the words from smallest to largest. The Nonmonotonic Numeric Pattern factor presents the student with an incomplete pattern of numbers in a combination of ascending and descending order. The student fills in the boxes with the missing numbers. Figure 1 displays examples of items for each of the four factors.

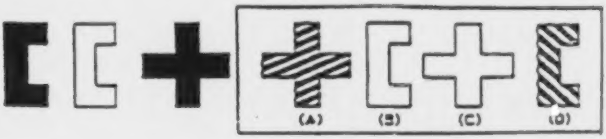
The items within each of the factors were combined to comprise four subscales. Cronbach's Alpha estimate of reliability was computed for each of the four subscales for both pretest and posttest data. The reliabilities range from 0.81 to 0.95. Descriptive statistics for each of the subscales and the total instrument appear in Table 1.

Table 2 presents the means and standard deviations for each of the subscales and the total instrument by grade by gender. This information is given for the pretest and posttest responses after pattern instruction. Analyses of variance indicates no significant treatment group by grade level by gender interaction effects.

The only significant two-way effect is a grade by gender interaction on the posttest for the Figural Pattern subscale, $F(1, 174) = 4.01$, $MSE = 7.42$, $p \leq .05$. The fourth grade boys ($M = 11.49$) scored higher than the fifth grade boys ($M = 9.88$). The fourth grade girls ($M = 10.80$) and the fifth grade girls ($M = 11.04$) were nearly equivalent although the fifth grade girls scored slightly higher than the fourth grade girls. The sample of fourth grade boys in this study was par-

**Factor 1:** Figural Pattern (FP)

FP03 What comes next?



**Factor 2:** Monotonic Numeric Pattern (MNP)

MNP01 Fill in the boxes with missing numbers.

9, 12, ☐, 18, 21, ☐

**Factor 3:** Word Pattern (WP)

WP04 Order from smallest to largest.

minute, second, hour ______, ______, ______

**Factor 4:** Nonmonotonic Numeric Pattern (NMNP)

NMNP05 & NMNP06 Fill in the boxes with the missing numbers.

☐, ☐, 21, 24, 20, 25, 19, 26

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ticularly adept on the posttest Figural Pattern tasks.

On the pretest Figural Pattern subscale, the pattern of scores for the boys is similar to that of the posttest. The fourth grade boys ($M$ = 11.40) outperformed the fifth grade boys ($M$ = 8.85).

The girls exhibit the same pretest pattern as the boys. The fourth grade girls ($M$ = 10.90) outperformed the fifth grade girls ($M$ = 9.67). It appears that the fourth grade students initially outperformed the fifth grade students. Whereas this pattern continues on the posttest for the boys, a change in pattern occurs for the girls. After instruction, the fifth grade girls ($M$ = 11.04) slightly outperformed the fourth grade girls ($M$ = 10.80) on the posttest.

Analyses of variance for Grade effect revealed significant multivariate, $F$ (8, 342) = 7.83, $p \leq .00$ and univariate, $F$ (2, 174) = 19.17, $MSE$ = 7.67, $p \leq .00$,

**Table 1** *Descriptive Statistics for the Spatial-Symbolic Pattern Instrument*

| | Subscale Name | | | | |
|---|---|---|---|---|---|
| | Figural Pattern (FP) | Monotonic Numeric Pattern (MNP) | Word Pattern (WP) | Nonmonotonic Numeric Pattern (NMNP) | Total |
| **Instrument** | | | | | |
| # of Items | 13 | 9 | 5 | 6 | 33 |
| Maximum | 13 | 9 | 15 | 6 | 43 |
| **Pretest** | | | | | |
| Reliabilities | 0.81 | 0.83 | 0.95 | 0.83 | 0.89 |
| % Variance | 16.03 | 6.70 | 4.86 | 4.43 | 32.02 |
| *M* | 10.24 | 4.74 | 12.94 | 3.07 | 30.98 |
| *SD* | 2.88 | 2.72 | 4.38 | 2.05 | 8.36 |
| **Posttest** | | | | | |
| Reliabilities | 0.84 | 0.84 | 0.94 | 0.82 | 0.89 |
| % Variance | 16.98 | 7.42 | 5.18 | 4.11 | 33.69 |
| *M* | 10.82 | 5.16 | 13.69 | 3.51 | 33.10 |
| *SD* | 2.73 | 2.77 | 3.56 | 2.06 | 7.28 |

**Table 2** *Means and Standard Deviations for the Spatial-Symbolic Pattern Instrument by Grade by Gender*

| | Pretest | | | | Posttest | | | |
|---|---|---|---|---|---|---|---|---|
| | 4th | | 5th | | 4th | | 5th | |
| | Boys | Girls | Boys | Girls | Boys | Girls | Boys | Girls |
| **FP** | | | | | | | | |
| *M* | 11.40 | 10.90 | 8.85 | 9.67 | 11.49 | 10.80 | 9.88 | 11.04 |
| *SD* | 1.51 | 2.67 | 3.16 | 3.19 | 1.64 | 2.89 | 3.41 | 2.54 |
| **MNP** | | | | | | | | |
| *M* | 4.70 | 4.60 | 4.61 | 4.94 | 4.79 | 5.00 | 5.02 | 5.71 |
| *SD* | 2.34 | 2.73 | 2.93 | 2.89 | 2.66 | 2.87 | 2.87 | 2.70 |
| **WP** | | | | | | | | |
| *M* | 12.81 | 13.94 | 11.98 | 12.77 | 11.49 | 11.22 | 14.10 | 14.00 |
| *SD* | 4.78 | 2.96 | 5.15 | 4.45 | 4.02 | 4.10 | 3.14 | 3.03 |
| **NMNP** | | | | | | | | |
| *M* | 3.09 | 3.16 | 2.85 | 3.12 | 3.42 | 3.42 | 3.42 | 3.73 |
| *SD* | 1.74 | 2.10 | 2.16 | 2.18 | 1.78 | 2.19 | 2.16 | 2.11 |
| Total | | | | | | | | |
| *M* | 32.00 | 32.60 | 28.32 | 30.50 | 33.19 | 32.44 | 32.42 | 34.48 |
| *SD* | 7.33 | 7.77 | 9.39 | 8.58 | 6.71 | 7.80 | 7.91 | 6.70 |

differences for the pretest Figural Pattern subscale. The fourth grade boys and girls combined (*M* = 11.13) scored significantly higher than the fifth grade boys and girls combined (*M* = 9.31). The fourth grade pretest scores were close to the maximum possible score of 13. As such, there was little room for improvement after instruction as measured by the posttest. The fifth grade pretest scores were nearly two points lower. Consequently, after instruction their scores increased bringing the pretest and posttest scores closer together which may explain the lack of a significant grade level effect for the posttest Figural Pattern subscale.

## Conclusions and Implications

Previous analyses of the instrument (N=165) revealed no grade by gender and no gender differences for any of the subscales (Berlin & White, 1992). In the current study, grade by gender differences were found for the posttest Figural Pattern subscale. Consistent across the two studies is the significant grade level effect for the Figural Pattern subscale, but in the current study the fourth graders unexpectedly outperformed the fifth graders on the pretest Figural Pattern subscale. In Berlin and White (1992), significant effects for grade level were found for two of the three subscales: Figural Pattern and Numeric Pattern. These grade level differences were not unexpected. Student's scores for the Figural Pattern subscale indicated significant differences between the third grade students (*M* = 11.7) and both the fourth (*M* = 13.0) and fifth grade (*M* = 13.5) students. For the Numeric Pattern subscale, students scores significantly increased with advancement from third (*M* = 6.8) to fourth (*M* = 8.7) to fifth grade (*M* = 11.2).

Based on the two analyses of the Spatial-Symbolic Pattern Instrument, we conclude that:

1. The original 57 items of the Spatial-Symbolic Pattern Instrument can be factored into four nearly independent factors, i.e., Figural Pattern, Monotonic Numeric Pattern, Word Pattern, and Nonmonotonic Numeric Pattern, accounting for approximately 33% of the original variance.
2. Four reliable subscales of the Spatial-Symbolic Pattern Instrument, and in particular, the Figural Pattern subscale, detect differences in individual and group ability to recognize and extend patterns. The four subscales can also reflect changes in performance after instruction.
3. In the present study, reasons for fourth graders outperforming fifth graders on the pretest Figural Pattern subscale are not evident. Perhaps the current fourth grade teacher provided more pattern activities than the current fifth grade teacher or the previous fourth grade teacher.
4. Further refinement of the Spatial-Symbolic Pattern Instrument might include more items of a similar nature to increase the reliability and

items of higher difficulty to improve the discrimination at the higher ability levels.

5. Further validation of the Spatial-Symbolic Pattern Instrument across grades one through six to represent a broader range of pattern recognition and extension abilities is needed.

## References

American Association for the Advancement of Science. (1989). *Project 2061, Science for all Americans.* Washington, DC: Author.

Battista, M. T., & Clements, D. H. (1996). Student's understanding of three-dimensional rectangular arrays of cubes. *Journal for Research in Mathematics Education, 27*(3), 258-292.

Berlin, D. F., & White, A. L. (1992). Validation of an instrument to measure understanding of pattern by elementary school children. *Perceptual and Motor Skills, 74*, 1225-1226.

Bruni, J. V., & Seidenstein, R. B. (1990). Geometric concepts and spatial sense. In J. N. Payne (Ed.), *Mathematics for the young child* (pp. 202-227). Reston, VA: National Council of Teachers of Mathematics.

Liedtke, W. W. (1995). Developing spatial abilities in the early grades. *Teaching Children Mathematics, 2*(1), 12-18.

McGee, M. G. (1979). *Human spatial abilities. Sources of sex differences.* New York: Praeger Publishers.

National Research Council. (1990). *Reshaping school mathematics: A philosophy and framework for curriculum.* Washington, DC: National Academy Press.

Piaget, J., & Inhelder, B. (1967). *The child's conception of space.* New York: W. W. Norton.

Small, M. Y., & Morton, M. E. (1983, Sept./Oct.). Spatial visualization training improves performance in organic chemistry. *Journal of College Science Teaching*, 41-43.

Steen, L. A. (Ed.). (1990). *On the shoulders of giants: New approaches to numeracy.* Washington, DC: National Academy Press.

von Glasersfeld, E. (1982). Subitizing: The role of figural patterns in the development of numerical concepts. *Archives de Psychologie, 50*, 191-218.

Wheatley, G. H. (1990). Spatial sense and mathematical learning. *Arithmetic Teacher, 37*, 10-11.

Wheatley, G. H. (Ed.). (1991). Enhancing mathematics learning through imagery. *Arithmetic Teacher, 97*, 34-36.

# VIEWS ABOUT MATHEMATICS SURVEY: DESIGN AND RESULTS

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The mathematical views of two populations of university students are reported. The Views About Mathematics Survey (VAMS), an instrument to assess and characterize student views about knowing and learning mathematics, was designed and administered to both precalculus and third semester calculus students. Student views are portrayed for both populations and are compared across course, gender, and course grade. Results reveal that: (a) undergraduate students hold views about knowing and learning mathematics that often diverge from the views of mathematicians and educators; (b) precalculus students' views are not noticeably affected by moderately reformed mathematics instruction, but (c) they do affect how students perform during the course; (d) gender differences were observed with male students reporting greater persistence and confidence than female students; and finally, (e) confidence correlates significantly with achievement and aspects of expert mathematical beliefs.

## Introduction

Most teachers would agree that a student's ability to respond to a problem situation involves much more than mathematical knowledge. A student who is unwilling to persist, has little confidence, and makes poor decisions regarding his/her solution approach is less likely to successfully complete a solution to a complex problem than a confident student who persistently invests time in a meaningful way (Schoenfeld, 1989; Lester et. al, 1989). Even though different students may have the same mathematical knowledge, their differing views appear to have a tremendous impact on students' success continued study of mathematics (Carlson, in press). These observations support the view that students' beliefs are an important aspect of students' mathematical development, yet research on affect in mathematics education continues to reside on the periphery of the field (McLeod, 1995). Even though current reform documents place increased emphasis on the role of affect, educational researchers continue to observe that students at all levels hold views about mathematics that are opposed to the views they are expected to develop in a mathematics courses (McLeod, 1995).

**Research reports:**

- Consistent trends in the correlation between achievement and confidence in learning and doing mathematics (Fennema and Sherman, 1977; Meyer and Koehler; 1990; Reyes, 1984; Schoenfeld, 1989; Lester et. al., 1989, Tartre and Fennema, 1995).

- Positive correlation between achievement, and perceived usefulness, motivation and perceived personal control (Fennema & Sherman, 1976);
- Persistence is a valuable trait, but only if accompanied by appropriate monitoring behaviors (Schoenfeld, 1989; Lester et. al., 1989);
- Students believe mathematics involves mainly memorization (Dossey et. al., 1988; Schoenfeld, 1989).

These studies suggest aspects of student beliefs worthy of investigation. To broadly characterize and quantify student views about the methods, learnability, and relevance of mathematics, an instrument was needed that could be widely administered to large populations. The Views About Mathematics Survey (VAMS) was developed for this purpose.

**The Taxonomy and Instrument**

VAMS was designed to assess students' views about knowing and learning mathematics. Its design followed that of the widely used VASS (Views About Science Survey) developed by Halloun & Hestenes (1996) and was developed cooperatively with the developers of this instrument. More specifically, VAMS was designed to:

- Discern significant differences among the views of mathematics students, mathematics teachers, and mathematicians.
- Identify patterns in student views and classify them in general profiles.
- Compare student views/profiles at various grade levels (8-16).
- Assess the relationship between student views/profiles and achievement.
- Measure the effectiveness of instruction in changing student views and profiles.
- Compare student views/profiles across various demographic strata.

A taxonomy of six major epistemological and pedagogical principles provides the theoretical model for the instrument's design. The development of this taxonomy emerged concurrently with the development of individual VAMS items, involving repeated refinement of the six dimensions, utilizing peer reviews, preliminary data analysis of early student responses to VAMS items and subsequent student interviews regarding students' interpretations of VAMS items. As a result, individual items were validated and three epistemological and three pedagogical taxonomy dimensions emerged. The epistemological dimensions pertain to: the structure of mathematical knowledge; the validity of mathematical knowledge; and the methods of mathematics. Pedagogical dimensions pertain to: learnability of mathematics; the role of critical thinking; and personal relevance of mathematics.

Each of the six dimensions is presented below in the form of a set of contrasting views about knowing and learning mathematics. The primary view or expert view corresponds to the views most commonly held among mathematicians. This view was established by administering the survey to a collection of mathematicians. The secondary view or folk view is the contrasting view often attributed to the lay community and naive student of mathematics.

### VAMS Taxonomy

1. **Structure**: Mathematics is a coherent body of knowledge about relationships and patterns contrived by careful investigation-rather than a collection of isolated facts and algorithms.

2. **Methodology**: The methods of mathematics are systematic and generic-rather than idiosyncratic and situation specific.

   Mathematical modeling for problem solving involves more-than selecting formulas for number crunching.

   Mathematicians use technology more to enhance their ways of solving problems-than to allow them to get quick easy solutions.

3. **Validity**: Mathematical knowledge is validated by logical proofs-rather than by correspondence to the real world.

   Mathematical knowledge is tentative and refutable-rather than absolute and final.

4. **Learnability**: Mathematics is learnable by anyone willing to make the effort-rather than by a few talented people.

   Achievement depends more on persistent effort-than on the influence of teacher or textbook.

5. **Critical Thinking**: For meaningful understanding of mathematics, one needs to:

   a) concentrate more on the systematic use of general thought processes-than on memorizing isolated facts and algorithms.

   b) examine situations in many ways, and not feel intimidated by committing mistakes-rather than follow a single approach from an authoritative source.

   c) look for discrepancies in one's own knowledge-instead of just accumulating new information;

   d) reconstruct new knowledge in one's own way-instead of memorizing it as given.

6. **Personal relevance**: Mathematics and related technology are relevant to everyone's life-rather than being of exclusive concern to mathematicians.

   Mathematics should be studied more for personal benefit-than for just fulfilling curriculum requirements.

Individual survey items consist of a statement followed by two contrasting alternatives which respondents are asked to balance on an eight-point scale (Table 1). Respondents can select either a weighted combination of the two alternatives (options 2, 3, 4, 5 or 6), either alternative (options 1 or 7), or neither of the two alternatives (option 8). By providing two clearly stated benchmarks, ambiguity among student responses is reduced and reliability for specific items is increased. This design, know as the Contrasting Alternative Design (CAD) produces instruments which are more valid, reliable and applicable to large populations (Halloun and Hestenes, 1996).

VAMS items measure students' views concerning the taxonomy items: structure of mathematics; methods of mathematics (Example 1, Example 2); learnability of mathematics (Example 3); role of critical thinking in doing mathematics (Example 4), relevance of mathematics (Example 5); and validity of mathematics. A response choice diagram and five VAMS items are presented in Figure 1.

The survey was revised based on peer reviews and student interviews. Interviews were conducted with approximately twenty-five students, securing a sufficient sampling from each of low, average and superior class performers. Each interviewee was asked to verbally respond to select survey items without looking at her/his earlier provided written response. Consistent responses were noted when comparing students' written and verbal responses. Instrument validity was assessed by asking students to justify selected responses. Consistency between the interviewees' responses and the design intent was analyzed. Where inconsistencies occurred, items were rewritten. This process of instrument refinement continues.

VAMS was administered to a broad community of mathematicians and college instructors in order to: (a) establish baseline data for experts, and (b) compare students' views with those of the experts. As a result of analyzing the expert and student data, students were classified to four broad groups of distinct profiles (expert, high transitional, low transitional and folk) using the following criteria:

- A student should belong to exactly one profile.
- A profile should consist of relatively coherent views. (e.g., A student with an expert profile should express more expert views than students having either an upper transitional, lower transitional or folk profile.)

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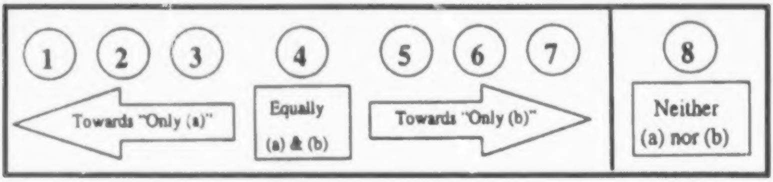
**Table 1** *A Response Choice Diagram*

1. For me, making unsuccessful attempts when solving a mathematics problem is:
   (a) a natural part of my pursuit of a solution to the problem.
   (b) an indication of my incompetence in mathematics.
2. In solving mathematics problems, graphing calculators or computers help me:
   (a) understand the underlying mathematical ideas.
   (b) obtain numerical answers to problems.
3. When I experience a difficulty while studying mathematics:
   (a) I immediately seek help, or give up trying.
   (b) I try hard to figure it out on my own.
4. My score on a mathematics exam is a measure of how well:
   (a) I understand the covered material.
   (b) I can do things the way they are done by the teacher or in course materials.
5. I study mathematics:
   (a) to satisfy course requirements.
   (b) to learn useful knowledge.

- The threshold for the expert profile should be such that no student can be classified as expert if he/she expresses expert views on less items, or folk views on more items, than a mathematician does.

## The Subjects, Procedures and Data Analysis

The subjects for this study were selected from two different levels of mathematical preparation, precalculus and third semester calculus. The precalculus students were taught in small sections of class sizes ranging from 30 to 40 students. The curriculum included an early introduction to functions with emphasis on applied problems and full integration of graphing calculators in both the class presentation and student activities. The third semester calculus students were taught in class sizes ranging from 35 to 50, using a reform text with lecture the primary mode of instruction and some integration of technology.



1. For me, making unsuccessful attempts when solving a mathematics problem is:

   (a) A natural part of my pursuit of a solution to the problem.

   (b) An indication of my incompetence in mathematics.

2. In solving mathematics problems, graphing calculators or computers help me:

   (a) Understand the underlying mathematical ideas.

   (b) Obtain numerical answers to problems.

3. When I experience a difficulty while studying mathematics:

   (a) I immediately seek help, or give up trying.

   (b) I try hard to figure it out on my own.

4. My score on a mathematics exam is a measure of how well:

   (a) I understand the covered material.

   (b) I can do things the way they are done by the teachers or in course materials

5. I study mathematics:

   (a) To satisfy course requirements.

   (b) To learn useful knowledge.

**Figure 1** *A response choice diagram*

There were approximately 550 precalculus students who completed the survey early in the semester and again during the last week of the semester. Also, there were 73 third semester calculus students who completed the survey once, at the beginning of the semester. Students responded to survey items on a bubble answer sheet after receiving brief instructions for completing the demographic items and a sample survey item.

## Results

The analysis of VAMS data included comparisons of: views across gender; student views with course performance, pre- and postVAMS responses; and cofidence with performance profile. The following conclusions resulted from these analyses.

- Precalculus and third semester calculus students hold views about knowing and learning mathematics that are incompatible with views commonly held in the mathematical and educational communities. 72% of precalculus and 51% of calculus students were classified with either a folk or lower transitional profile.
- Student views regarding the learnability and methods of mathematics correlate positively with mathematics achievement. "A" students tended more toward the expert view and "D" and "F" students tended more toward the folk view on the majority of VAMS items. Of those students possessing an expert profile, 88% received either an "A" or "B". The higher performing students (i.e., A and B) report believing that: meaningful understanding of mathematics requires reconstruction of knowledge instead of memorizing it as given; the methods of mathematics are more systematic and generic than situation specific; and mathematics should be studied more for personal benefit than fulfilling curriculum requirements.
- Consistent with the results of Fennema and Sherman (1978), mathematical confidence was found to correlate positively with mathematical achievement. Additionally, those students reporting greater confidence were more likely to be male; receive higher grades; continue their study of mathematics; persist when confronting a difficulty while studying mathematics; view unsuccessful attempts as a natural part of the pursuit of a problem's solution; and understand that methods for problem solving are more generic and systematic as opposed to situation specific.
- Differences in the views of males and females were observed. Among the most notable was the way males and females approach challenging mathematics problems. For the 355 female and 265 male surveyed participants, we observed a higher proportion of females reporting frustration when solving challenging problems and a higher proportion of males reporting greater confidence, enjoyment and persistence when confronting a challenging problem.
- College students' views do not appear to change over the course of one semester, even with modest attempts to deliver reform instruction. Very little difference exists in the response patterns between the pre and post

VAMS results on the 27 VAMS items. A very high proportion of precalculus students report non-persistence when pursuing a problem's solution and believe that successful performance is more dependent on imitating others' solutions rather than applying general problem solving approaches.

- Differences exist in the distribution of profiles for precalculus and third semester calculus students. Precalculus students have proportionately higher categorization into the folk or lower transitional profiles while calculus students have fairly equal distributions into each of the four profiles. This difference in distribution could suggest that students' views do change gradually with continued mathematical instruction.

## Acknowledgments

The authors would like to thank Dr. Ibrahim Halloun and Dr. David Hestenes for their guidance in the development of the VAMS instrument. We would also like to thank Melissa Fazzari for her assistance in the collection and analysis of data.

## References

Armstrong, J.M. (1985). A national assessment of participation and achievement of women in mathematics. In *Women in Mathematics: Balancing the Equation* (pp. 59-94). Hillsdale, NJ: Erlbaum.

Carlson, M. (in press). A cross-sectional investigation of the development of the function concept. *Research in Collegiate Mathematics Education.*

Carpenter, T. P., Lindquist, M. M., Matthews, W., & Silver, E. A. (1983). Results of the third NAEP mathematics assessment: Secondary school. *Mathematics Teacher, 76*(9), 652-659.

Crosswhite, F. J. (1972). *Correlates of attitudes toward mathematics, National Longitudinal Study of Mathematical Abilities, Report No. 20.* Palo Alto, CA: Stanford University Press.

Fennema, E., & Sherman, J. A. (1976). Fennema-Sherman Mathematics Attitude Scales: Instruments designed to measure attitudes toward the learning of mathematics by females and males. *Journal for Research in Mathematics Education, 7*(4), 324-326.

Fennema, E., & Sherman, J. A. (1977). Sex-related differences in mathematics achievement, spatial visualization and affective factors. *American Education Research Journal, 14*(1), 51-71.

Fennema, E., & Sherman, J. A. (1978). Sex-related differences in mathematical achievement and related factors: A further study. *Journal for Research in Mathematics Education, 9,* 189-203.

Garofalo, J. (1989). Beliefs, responses, and mathematics education: Observations from the back of the classroom. *School Science and Mathematics, 89,* 451-455.

Halloun, I., Hestenes D. (1996). *Views about sciences Survey*. NARST Annual Meeting, St. Louis, Missouri.

Lester, F. K., Garofalo, J., & Kroll, D. L. (1989). Self-confidence, interest, beliefs, and metacognition: Key influences on problem-solving behavior. *Affect and Mathematical Problem Solving: A New Perspective* (pp. 75-88). New York, NY: Springer-Verlag.

McLeod, D. B., & Adams, V. M. (1989). *Affect and mathematical problem solving: A new perspective.* New York, NY: Springer-Verlag.

Meyer, M.R. and M.S. Koehler (1990). Internal influences on gender differences in mathematics, In Fennema & Leder, (Eds.). *Mathematics and gender: Influences on teachers and students* (pp. 60-95). New York, NY: Teachers' College Press.

Reyes, L. (1984). Affective variables and mathematics education. *The Elementary School Journal, 84*(5), pp. 558-581.

Schoenfeld, A. H. (1988). When good teaching leads to bad results: The disasters of "well-taught" mathematics courses. *Educational Psychologist, 23,* 145-166.

Schoenfeld, A. H. (1989). Explorations of students' mathematical beliefs and behavior. *Journal of Research in Mathematics Education, 20,* 338-355.

# COLLEGE STUDENTS' SELF-MOTIVATION AND MATHEMATICAL BELIEF STRUCTURE

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The purpose of this study was to examine and describe student beliefs about oneself as learners of mathematics and beliefs about mathematics and to relate these beliefs to prior exposure to mathematics and to achievement. Based on data in a self-report questionnaire from 266 students starting a study program in economics and business administration, we identified eight belief factors with a total of 25 indicators relating to the variables: self-efficacy for self-regulated learning, self-efficacy as part of motivational beliefs, student self-conception, mathematics as an interesting subject, mathematics anxiety, understanding concepts, mathematics as a useful subject. To study the relationships between our constructs, we applied a structural equation model approach with latent variables. A confirmatory factor analysis based on a measurement model with the belief factors as latent variables, indicated that the given set of indicators gave an adequate description of students beliefs. Following social cognitive theories we investigated a structural model with self-efficacy beliefs playing a mediating role in the influence of prior exposure in mathematics on students later beliefs and achievement. We found that the variables for self-efficacy, self-concept and interest correlated strongly with prior exposure to mathematics and the achievement test result correlated moderately with the belief factors. Our study indicate self-efficacy, self-concept, interest and anxiety as important variables related to student learning in mathematics.

## References

Bandura, A. (Ed.). (1995). *Self-Efficacy in Changing Societies.* Cambridge: Cambridge University Press.

Carr, M. (Ed.). (1996). *Motivation in Mathematics.* New York: Hampton Press.

Pajares, F., & Miller, M. D. (1994). Role of Self-Efficacy and Self-Concept Beliefs in Mathematical Problem Solving: A Path Analysis. *Journal of Educational Psychology, 86*(2), 193-203.

Pintrich P. R., & De Groot, E. V. (1990). Motivational and Self-Regulated Learning Components of Classroom Academic Performance. *Journal of Educational Psychology, 82*(1), 33-40.

# CARICATURES VERSUS CASE STUDIES: A DISCUSSION OF TWO METHODS OF RESEARCH REPORTING

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Attempts to relate qualitative research should constantly undergo scrutiny as researchers try to most effectively communicate findings and analyses. Research reporting needs to be cogent, clear, concise, compelling, and complete. At times, these characteristics seem to be at cross-purposes from each other. For example, completeness often works against conciseness. Apparently completeness is winning the battle with conciseness — witness the recent trend in increased length of research reports (30% over five years) submitted to JRME (Lester & Sowder, 1996). One way to achieve conciseness in research reports is through the use of caricatures.

We use caricatures in our title because we have used this term in our research reporting (Lambdin & Preston, 1995). However, perhaps even our so-called caricatures are more like composite profiles, where we took traits and tales from teachers and funneled them into a single portrait. This funneling is typically used for one of three reasons: confidentiality, data compression, or to provide distinct ideal types. Pitfalls to watch for with caricatures include loss of the idiosyncratic voice, danger of stereotyping, and difficulty in obtaining trustworthiness through a member check. For the sake of brevity, we assume that participants are aware of pros and cons in reporting cases, a technique we also use (e.g., Preston & Lambdin, in press).

The purpose of this discussion is to elicit the opinions of participants in using caricatures or cases to report research and to tie this discussion to issues such as length of articles, confidentiality, theme unification, loss of idiosyncratic voice, trustworthiness, and stereotyping. Participants may pick up a paper at PME-NA giving more detailed background on caricatures and cases in advance of the session. This paper also contains questions for discussion.

## References

Lambdin, D. V., & Preston, R. V. (1995). Caricatures in innovation: Teacher adaptation to an investigation-oriented middle school mathematics curriculum. *Journal of Teacher Education, 46*, 130-140.

Lester, F. K., & Sowder, J. T. (1996). Editorial. *Journal for Research in Mathematics Education, 27*, 514-515.

Preston, R. V., & Lambdin, D. V. (in press). Mathematics for all teachers? Factors that hinder teachers' efforts to implement mathematics education reforms. *The Journal of Mathematics Teacher Education.*

# TEACHER BELIEFS

# CHANGES IN TEACHERS' BELIEFS AND ASSESSMENTS OF STUDENTS' THINKING ACROSS THE FIRST YEAR OF IMPLEMENTATION OF COGNITIVELY GUIDED INSTRUCTION

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At the beginning of May 1995 and June 1996 workshops, 21 female teachers in grades K-5, who were in their first year of a five-year CGI project, completed (a) a Transcript Analysis of a dialogue between a first-grade teacher and three students and (b) a 48-item Beliefs Scale. Prior to each workshop, participants had responded to two General Items concerning how teachers know what students understand. Responses to the General Items and the Transcript Analysis were categorized and analyzed within and across instruments. Pre/post responses for the General Items and the Transcript Analysis were contrasted with changes noted in subscales of the Beliefs Scale. Teachers beliefs changed significantly in ways that were consistent with the major tenets of CGI. Similarly, the evidence they cited to support their assessment of students' thinking also changed in ways consistent with the implementation of CGI. However, complex relationships appear to exist between these two kinds of changes.

The data for this study were gathered at the beginning and end of the first year of a five-year teacher enhancement project (NSF Grant ESI-09450518) in which elementary teachers are being given opportunities to learn to use cognitively guided instruction (CGI) as a basis of mathematics instruction. A variety of longitudinal data are being gathered on teachers' beliefs, interpretations of children's solutions to mathematics problems, and instructional decision making. This paper is a report of teachers' beliefs and their interpretations of children's problem solving performance.

CGI is an approach to teaching mathematics in which knowledge of children's thinking is central to instructional decision making. Teachers use research-based knowledge about children's mathematical thinking to help them learn specifics about individual students and then to adjust instruction (e.g., sequencing of problems, kinds of numbers used in problems) to match students' performance. In implementing CGI, teachers learn to assess students' thinking (primarily through listening to students explain solutions to mathematics problems) and then use that knowledge to plan instruction (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989).

## Method

Subjects were 21 female elementary teachers for whom there were complete baseline and end-of-first-year data. The Transcript Analysis and Beliefs Scale were administered on the first day of the initial three-day workshop in May 1995 and the summer workshop in June 1996. Subjects completed the General Items individually prior to each workshop.

*General Items.* Subjects responded in writing to two items.

1. What do you look for as you watch children solving mathematics problems? Why are those things important to you?
2. How do you know whether students understand mathematical ideas?

*Transcript Analysis.* The instrument contains a transcript of three teacher-and-student dialogues (Mac, Tom, and Sue) that occurred while a group of 23 first-grade students worked individually on 5 written problems. The teacher interacted with Mac after he had completed the problem: *If frog's sandwiches cost 10 cents, and he had 15 sandwiches, how much did frog's sandwiches cost altogether?* As the teacher moved to Tom's desk, Tom was working on the same problem. The teacher's interaction with Sue occurred as she was working on a different problem: *Frog had 15 sandwiches. If each sandwich cost 5 cents, how much do all the sandwiches cost altogether?*

After reading the transcript, subjects were asked to state their conclusions about the three children's (a) levels of thinking and (b) mathematical understanding. Subjects were also asked to identify specific evidence from the transcript that was important to them in reaching those conclusions. No definition for the phrases "levels of thinking" or "mathematical understanding" were requested or provided during the administration of this instrument. Earlier we reported subjects' initial interpretations of the transcript (Bowman, Bright, & Vacc, 1996).

*Beliefs Scale.* The Beliefs Scale (Peterson, Fennema, Carpenter & Loef, 1989) has four subscales: Role of the Learner, Relationship Between Skills and Understanding, Sequencing of Topics, and Role of the Teacher. Each subscale contains 12 items, each of which is rated on a five-point Likert scale. Higher scores indicate beliefs that are more consistent with constructivism. Internal consistency estimates for the total score have been reported as .93, while estimates for each subscale range from .57 to .86 (Peterson, et al., 1989).

*Analysis of Responses.* Content analysis on verbatim written responses for each open-ended item was completed manually. Responses were dissected, fragments were grouped by content, and category labels were identified for clusters of comments. Then evidence for each category was discussed by the authors, until agreement was reached on the nature of evidence that would be accepted for categorizing responses according to these frameworks. Finally

| Category | Representative Responses |
|---|---|
| 1. General perception of understanding | I ask [students] ... to check for understanding and reasoning skills. |
| 2. Use of problem solving processes | Orderly sequence of steps |
| 3. Multiple or unique solutions to problems | I ask them to solve the problem another way (after they show one way).<br>I also look to see if the child is developing a strategy of his/her own to solve the problem. |
| 4. Explanation or demonstration by students | I get the children to talk so I can clear up any confusion.<br>Listening to their reasoning processes. |
| 5. Correctness of answers | I look to see ... if the answer is correct. |
| 6. Use of prerequisite skills or knowledge | Do they know number facts? |
| 7. Application of knowledge to other situations | I look to see if students are applying what has been taught. |
| 8. Time spent on problem | If students ... solve problems quickly. |
| 9. Student involvement, affect | I look for their willingness to try. I watch for frustration. |
| 10. Methods of assessment | By providing many opportunities for the children to demonstrate his/her mathematical thinking. |

**Figure 1** *Categories of evidence cited by subjects in their responses to the General Items*

data were coded independently by the authors, and codings were compared and negotiated whenever necessary.

For the General Items, ten categories were identified; for the Transcript Analysis, seven categories were identified. These categories, along with brief illustrative quotes from subjects' responses, are presented in Figures 1 and 2. Data from the Beliefs Scale were summarized with descriptive statistics and repeated measures analysis.

## Results

*General Items.* For the first General Item, the categories appearing most often in the analysis (per cent of subjects for first administration/per cent of subjects for second administration) were "general perception of understanding" (57%/57%), "use of problem solving processes" (67%/71%), and "explanation or demonstration of solutions by students" (43%/57%). For the second General Item, the categories appearing most often were "use of problem solving processes" (19%/38%), "explanation or demonstration of solutions by students" (71%/86%), and "application of knowledge to other situations" (48%/19%; $t = -2.09$, $p < .05$).

*Transcript Analysis.* The categories appearing most often were "use of manipulatives, computation, tools" (95%/100%), "use of place value, money concepts and notation" (90%/95%), "teacher guidance" (71%/38%; $t = -2.28$, $p < .05$), "multiple solutions to problems" (43%/33%), and "explanation of solution by student" (33%/24%).

*Beliefs Scale.* Responses changed significantly toward a more constructivist perspective on all four subscales (maximum score of 60 on each subscale): Role of the Learner (42.8/48.8; $F(1,20) = 23.7$, $p < .0001$), Relationship Between Skills and Understanding (45.8/50.4; $F(1,20) = 22.8$, $p < .0001$), Sequencing of Topics (46.1/52.1; $F(1,20) = 33.0$, $p < .0001$), and Role of the Teacher (45.6/52.9; $F(1,20) = 44.8$, $p < .0001$).

*Comparison across instruments.* Categories from Figures 1 and 2 were compared to the four subscales of the Beliefs Scale. Categories 1, 3, and 4 seemed directly related to the subscales, Role of the Learner and Role of the Teacher. For the General Items and the Transcript Analysis, each teacher's responses were coded as illustrating (1) or not illustrating (0) that category. Thus, for each category there were four groups: 0/0, 1/0, 0/1, and 1/1, with each pair of symbols representing categorization of responses for the two administrations. For the subjects within each category, an average change score was computed for each of the two relevant subscales of the Beliefs Scale. These data are presented in Figure 3.

| Category | Representative Response |
|---|---|
| 1. General perception of understanding | Tom has ... solved the problem.... Sue does not understand the problem. |
| 2. Use of manipulatives, computation, tools | Mac ... use[d] a number chart. |
| 3. Multiple solutions to problems | [Mac] ... solved the problem in more than one way. |
| 4. Explanation of solution by student | Mac is also able to explain. |
| 5. Use of place value, money concepts and notation | They were able to use knowledge of money. |
| 6. Teacher guidance | Even with assistance from the teacher [Sue] needs several attempts. |
| 7. Student involvement, affect | [Sue is] very unsure of her answers. |

**Figure 2** Categories of evidence cited by subjects in their responses to the Transcript Analysis

## Discussion

*General Items.* Teachers' responses to the first item indicate a focus on "general perception of understanding," "use of problem solving processes," and "explanation or demonstration of solutions by students." Increases in the latter two categories suggest that the emphasis placed on more specific evidence of student understanding was greater in the second administration. For the second item the same trend toward increased focus on "use of problem solving processes" and "explanations or demonstrations of solutions by students" occurs. Decreased emphasis on "application of knowledge to other situations" may reflect more specific focus by teachers on attending to students as they work on assigned problems and less focus on general evidence of application.

*Transcript Analysis.* Across administrations there was an increased focus on more specific content related to the particular problem solving task (e.g., "use of manipulatives, computation, tools," "use of place value, money con-

| | Group 0/0 | | | Group 1/0 | | | Group 0/1 | | | Group 1/1 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Item | N | Avg.L | Avg.T | N | Avg.L | Avg.T | N | Avg.L | Avg.T | N | Avg.L | Avg.T |
| General Item 1 Category 1 | 6 | 7.7 | 9.0 | 3 | 3.0 | 3.0 | 3 | 0.0 | 2.7 | 9 | 8.0 | 9.2 |
| General Item 1 Category 3 | 15 | 5.5 | 7.1 | 3 | 5.3 | 8.0 | 3 | 9.3 | 7.7 | 0 | — | — |
| General Item 1 Category 4 | 6 | 7.3 | 8.8 | 3 | 9.3 | 9.7 | 6 | 8.0 | 9.2 | 6 | 1.2 | 2.7 |
| General Item 2 Category 1 | 17 | 6.5 | 7.7 | 0 | — | — | 4 | 4.3 | 5.8 | 0 | — | — |
| General Item 2 Category 3 | 19 | 6.4 | 7.4 | 1 | -2.0 | 2.0 | 1 | 7.0 | 11.0 | 0 | — | — |
| General Item 2 Category 4 | 0 | — | — | 3 | 7.7 | 8.0 | 6 | 6.7 | 7.5 | 12 | 5.3 | 7.1 |
| Transcript Category 1 | 0 | — | — | 0 | — | — | 1 | 11.0 | 9.0 | 20 | 5.8 | 7.3 |
| Transcript Category 3 | 13 | 6.9 | 7.3 | 6 | 5.0 | 5.8 | 1 | -1.0 | 11.0 | 1 | 8.0 | 13.0 |
| Transcript Category 4 | 3 | 2.3 | 7.7 | 10 | 7.4 | 6.5 | 3 | 4.7 | 5.3 | 5 | 6.4 | 10.0 |

N=number of subjects; Avg.L=average change in Role of Learner; Avg.T=average change in Role of Teacher

**Figure 3**. *Comparison of data across instruments*

cepts and notation") and a decreased focus on general attributes of student performance (e.g., "teacher guidance," "multiple solutions to problems," "explanation of solution by student"). This result suggests that teachers can learn to focus on more specific information in their assessment of students' understanding; this trend is consistent with results of Fennema, Carpenter, Franke, Levi, Jacobs and Empson (1996).

*Belief Scale*. While all shifts in scores are toward a more constructivist approach, the shifts on Role of the Learner and Role of the Teacher may be particularly interesting, given the focus of teachers on "explanation or demonstration of solutions" for both the General Items and the Transcript Analysis and "multiple solutions to problems" for the Transcript Analysis. Shifts in these two subscales seem particularly consistent with a CGI approach — an approach that stresses the teacher's role in attending to both students' problem solving processes and their demonstration and explanation of solutions.

*Comparisons of data*. For all but three cells in Figure 3, the average change score for the Role of the Teacher is equal to or greater than the average change score for Role of the Student. These three exceptions are for subjects whose responses were categorized differently across time (i.e., the 1/0 or 0/1 group).

Because there are 12 cells (out of 36 total cells) in Figure 3 which contain data for either 1 or 0 subjects, it is difficult to see clear patterns by comparing corresponding cells. There are, however, some patterns in the data for category 4; in this category there is only one empty cell.

For category 4, for both subscales and for all three items, the change scores for the 1/0 subjects were greater than the change scores for the 0/1 subjects. Further, the change scores on Role of the Learner were greater for the 1/0 subjects than for all other subgroups. This is consistent with the notion that there might be a "threshold" of change, especially for Role of the Learner, such that if teachers exceed this threshold the focus on student explanation may become "routine" and "not worth mentioning" in a discussion of students' understanding of mathematics.

In category 4, for the 1/1 cells, the change scores for each subscale increased across items, while for all other cells, the change scores for each subscale decreased across items. That is, the beliefs of subjects in the 1/1 group for the Transcript Analysis changed more toward a constructivist perspective than the beliefs of subjects in the 1/1 group for the first General Item. In contrast, the beliefs of subjects in the other three subgroups for the Transcript Analysis changed less toward a constructivist perspective that the beliefs of subjects in similar subgroups for the first General Item. This suggests that what appears on the surface to be a similar way of categorizing subjects in fact identifies different kinds of people. There should be further investigation of how the different items may "trigger" different kinds of responses across the items, with those responses in turn being categorized differently.

Within category 4 for the General Items, the change scores for both subscales were greater for the 1/0 and 0/1 subjects than for the 0/0 and 1/1 subjects, while for the Transcript Analysis the change scores seemed marginally higher for the 0/0 and 1/1 subjects than for the 1/0 and 0/1 subjects. This shift in ranking of the change scores across items further suggests that there may be important differences in the ways that the items elicit information from teachers. For instance, there might be an interaction between the wording of the items and the amount of change in beliefs about Role of the Student and Role of the Teacher. Different levels of change toward constructivism may sensitize teachers to different use of language in their responses to the items.

In conclusion, across the first year of this project, teachers' beliefs changed and the evidence they cited to explain students' mathematical understanding also seemed to change. However, the relationships between these two kinds of changes appear to be quite complex.

## References

Bowman, A. H., Bright, G. W., & Vacc, N. N. (1996). Teachers' assessment of students' levels of thinking and mathematical understanding. In E. Jakubowski, D. Watkins, & H. Biske (Eds.), *Proceedings of Eighteenth Annual Meeting, North American Chapter of the International Group for the Psychology of Mathematics Education* (volume 2, pp. 521-526), Panama City, FL: Florida State University.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C-P., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal, 26*, 499-531.

Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V. R., & Empson, S. B. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education, 27*, 404-434.

Peterson, P. L., Fennema, E., Carpenter, T. P., & Loef, M. (1989). Teachers' pedagogical content beliefs in mathematics. *Cognition and Instruction, 6*, 1-40.

# SUSTAINING CULTURES OF TEACHING FOR CONSTRUCTIVE MATHEMATICS EDUCATION

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This study investigated the long-term impact of four teachers' participation in a dialogic community. Changes in mathematics instruction as well as on-going professional experiences were investigated both for the individual impact on the teachers and their classroom mathematics instruction and for the influence of the dialogic community on the larger school culture. It was clear the influence of the dialogic community reached three of the four teachers' personal and professional lives and significantly affected how they valued and developed classroom climates reflective of practices within the dialogic community and maintained professional working relationships with one another. On-going support and resources for sustaining community and affecting cultural change within a school seem necessary for collective change to occur.

There has been much written about and a great deal of money allocated for programs engaging teachers in systemic reform efforts. Over half of the states in the United States have received millions of dollars of federal money for state-wide systemic initiatives for mathematics and science reform while state Title II Eisenhower monies have targeted mathematics and science education reform efforts. Within this political and economic climate, where expectations and visions of the future along with beliefs that students are not learning the mathematics and science they need to be successful in the 21st century, there has been increased interest in professional development of teachers. How do existing school structures encourage or undermine attempts to change approaches to teaching that are more facilitative of meaningful mathematics learning?

Hargreaves (1994) discusses cultures of teaching which traditionally have emphasized individualism, isolation, and privatism. He defines the teaching culture to be:

> [the] beliefs, values, habits and assumed ways of doing things among communities of teachers who have had to deal with similar demands and constraints over many years. . . . [The teaching culture] forms a framework for occupational learning. . . . Cultures of teaching help give meaning, support and identity to teachers and their work. . . . What goes on inside the teacher's classroom cannot be divorced from the relations that are forged outside it. (p. 165)

Important for developing a culture of change is teacher reflectivity and commitment to transformative education (Frankenstein, 1987). There has been much in the literature pertaining to teacher reflection although recent critiques (Smyth, 1992) have suggested that "reflective teaching is entering a phase . . . where it has become co-opted and institutionalized. Like most educational reform before it, it is being cast in the mold of the technological mind set and thus support(s) standard practice rather than challenge(s) it" (p. 275). Much effort to support teacher reflection has failed to consider the teaching culture in which the teacher works and the impact of teachers working within communities. How can we help teacher communities confront and reconstruct their own teaching to promote evolution of teaching cultures conducive to responsive and constructive mathematics teaching?

## Previous Research

During the 1995-1996 school year, we worked intensely with four elementary school teachers participating in a professional dialogic community. These teachers met biweekly during the Fall semester as well as participated in weekly individual interviews immediately following classroom observations of mathematics instruction. The investigation examined the complex interplay among the dialogic process and classroom practices and norms (Pourdavood & Fleener, 1996), the dialectic relationship between the evolution of the dialogic community and changes in sociocultural norms in the classroom to include more humanistic classroom practices (Pourdavood & Fleener, in press a), and individual teacher's changing beliefs and ability to engage in critical and narrative reflective activity as the dialogic community evolved (Pourdavood & Fleener, in press b). These different perspectives of the depth of the relationship between their experiences in the dialogic community and the impact their participation in the project seemed to have on their teaching suggest the importance of the dialogic process for critical action and change. The question remains, however, whether these changes were sustained, especially given the central role the researcher-facilitator played in their dialogic experiences.

This study was a follow-up study of the previous one, one year later. The primary question of this investigation was: How have mathematics instruction and professional relationships been influenced, in the long run, by teachers' experiences participating in a dialogic community?

## Theoretical Assumptions

The theoretical and philosophical bases of this investigation include the beliefs that 'reality' is a social and political construction (Berger & Luckmann, 1966), that our universe is participatory (Capra, 1983) implying that there is a biological basis for our interactions with and position in co-creating the universe (Bateson, 1972; Maturana & Varela, 1982), that there is a moral imperative for teachers to engage in liberatory education (Freire, 1970, 1973; Greene, 1988), that meaning and context are dialectically constructed through patterns of relationships, actions, and language (Lemke, 1995), and that teacher cultures are vital to teacher change and educational reform (Hargreaves, 1994). An important component of educational reform, in our mind, is Freire's notion of conscientization (Freire, 1970/1996) as a transformative process involving "moving from being uncritical to critical, from being ahistorical to . . . historical" (Grundy, 1987, p. 190). The question remains how to facilitate the emergence of teacher culture centered on dialogic communication and critical and reconstructive reflection for liberatory education and reform.

## Methods

There were four teachers who participated in the original investigation during the fall semester, 1995. Macy taught first grade and had been teaching for two years at the time of the original study. Julie had been teaching second grade for 18 years. Lucinda had been teaching elementary and junior high for 25 years, the last three in third grade. Elizabeth had been teaching 20 years, the last five teaching gifted and talented for grades one through four.

The second year follow-up included two scheduled meetings with each of the researchers, classroom observations, and e-mail correspondence. Three of the four teachers met with one of the researchers on a Saturday in December, 1996, for approximately three hours. An additional one-hour interview session and classroom observations occurred toward the end of the second year in early May, 1997. All sessions with the teachers were audiotaped and partially transcribed. Detailed notes of classroom observations were made and e-mail discussions were included as an additional data source. Prolonged engagement, persistent observation, peer debriefing, and member checks (Guba & Lincoln, 1989) were used to ensure credibility of the study. Dependability and confirmability were ensured both by triangulation of data sources and by having each of the two researchers conduct separate group interviews.

## Findings

### Facilitating Dialogic Teaching Cultures

Each of the three teachers who participated in the follow-up interviews during the second year felt their experience participating in the dialogic community was rejuvenating and propelled them toward more connected and meaningful professional development. Elizabeth, for example, during the time of the original study, noted the value of having the researcher-facilitator provide them with up to date professional readings. Since then, she has become more active in her professional education fraternity, has been elected president of that organization, and has been developing her own professional library of readings. Lucinda and Julie (who could not participate in the follow-up interviews), also increased their level of commitment in this organization and continue to share with Elizabeth while participating in that organization. Similarly, Lucinda and Elizabeth have both participated in a summer program with one of us, and both continue to communicate with us via the internet.

Each of these teachers, during the follow-up interviews, expressed the value they placed on continued professional growth, maintaining professional contacts and renewing their own teaching and energy. "You should constantly be working to improve, . . . looking at different things" (Macy, May, 1997) was a value each of the teachers expressed. They did critique, however, the difficulty with current school structures for facilitating meaningful dialogue and critical growth. Macy commented:

> The facilitator role, though, is so important because this year it's been extremely difficult for us to say, ok, let's set up a meeting and talk. . . . So it's like you have to have a person, I think, to set the meetings and have some purposes and give some direction so it's not just a gab session (Macy, May, 1997)

Even with a supportive administration such as theirs, it is difficult, they felt, to maintain the intensity and focus they had had while participating in the dialogic community the previous year. These teachers became more autonomous in seeking out opportunities for professional growth and utilized the internet to explore and maintain relationships facilitative of their continued conscientization of their own teaching.

### Influence on Mathematics Instruction

Each of the three teachers participating in the follow-up indicated increased emphasis on student opportunities to discuss in their classes. Elizabeth, for example, felt she had learned to listen to her students and is aware this year of continuing to develop her questioning and listening skills in her mathematics instruction. All three teachers had continued to include opportunities for their students to work in cooperative groups. Lucinda felt that this year she continued to explore and make sense of her own teaching in a much more deliberate way than she had in the past. Classroom observations supported the teachers' expressed value of student discussion, active listening, and probing questioning of student thinking. For example, during Lucinda's meeting time, students wrestled for half an hour trying to solve a student generated problem - one billion minus what is one? Lucinda orchestrated this lively and meaningful conversation.

## Implications and Discussion

It seems the teaching culture of these three teachers was both receptive to and influenced by our dialogic community. The teachers' participation in the original study was voluntary, their school administrators were supportive, and the parents in their community were, if not supportive, not antagonistic. Participation in the dialogic community provided the teachers with an experience of community, openness, and trust which they came to value. To the extent that they reflected on the importance of the dialogic community for their own growth, they became more aware of and interested in nurturing that kind of environment in their own classrooms.

The resistance of the existing school structure to nurture relationships among teachers was apparent, however, when the teachers were asked whether their continued discussions had expanded to include others within the school. While the continued influence of the dialogic community on their relationships was apparent, broadening of the community to influence the culture of the school toward more teacher dialogue and teacher initiated professional development was not. Sustaining teacher cultures of dialogue and growth seems to require continued and structural commitments to the provision of resources and facilitators willing to organize and initiate opportunities for teachers to share and grow together.

## References

Bateson, G. (1972). *Steps to an ecology of mind*. New York: Ballantine Books.

Berger, P., & Luckmann, T. (1966). *The social construction of reality*. New York: Doubleday.

Capra, F. (1983). *The turning point: Science, society and the rising culture*. New York: Bantam.

Frankenstein, M. (1987). Critical mathematics education: An application of Paulo Freire's epistemology. In I. Shor (Ed.), *Freire for the classroom: A sourcebook for liberatory teaching* (pp. 180-210). Portsmouth, NH: Boynton/Cook Publishers, Inc.

Freire, P. (1973). *Education for critical consciousness*. New York: Seabury.

Freire, P. (1970/1996). *Pedagogy of the oppressed*. New York: The Continuum Publishing Company.

Greene, M. (1988). *The dialectic of freedom*. New York: Teachers College Press.

Grundy, S. (1987). *Curriculum: Product or praxis?* New York: The Falmer Press.

Guba, E. G., & Lincoln, Y. S. (1989). *Fourth generation evaluation*. Newbury Park, CA: Sage, Inc.

Hargreaves, A. (1994). *Changing teachers, changing times: Teachers' work and culture in the postmodern age*. New York: Teachers College Press.

Lemke, J. L. (1995). *Textual politics: Discourse and social dynamics*. Bristol, PA: Taylor & Francis.

Maturana, H., & Varela, F. (1980). *Autopoiesis and cognition*. Boston: D. Reidel Publishing.

Pourdavood, R., & Fleener, M. J. (1996). Examining evolving teachers' beliefs and practices within a dialogic community. In E. Jakubowski, D. Watkins, & H. Biske (Eds.), *Proceedings of the Eighteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 370-375). Panama City, FL: PME-NA.

Pourdavood, R., & Fleener, M. J. (in press a). Impact of a dialogic community on the development of classroom sociocultural norms. *The Journal for a Just and Caring Education*.

Pourdavood, R., & Fleener, M. J. (in press b). The ecology of a dialogic community as a socially constructive process. *Teaching Education*.

Smyth, J. (1992). Teachers' work and the politics of reflection. *American Educational Research Journal, 29*, 267-300.

# THE EDUCATIONAL BENEFITS OF BEING A PARTICIPANT IN A RESEARCH STUDY: ONE PRESERVICE SECONDARY MATHEMATICS TEACHER'S EXPERIENCE

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This paper reports the experiences of one preservice secondary mathematics teacher who participated in a study that was designed to elicit preservice teachers' beliefs about mathematics and its teaching and to provide opportunities for preservice teachers to examine their beliefs. The preservice teachers were asked to discuss their responses to a list of similes for learning mathematics and being a mathematics teacher, to work together in a problem-solving context to solve routine and nonroutine calculus problems, and to reflect on their beliefs and actions. In this paper, I discuss how these activities encouraged one preservice teacher to examine and elucidate her beliefs about mathematics, its learning, and its teaching.

## Background and Purpose

The current mathematics education reform movement calls for fundamental changes in the teaching and learning of mathematics (NCTM, 1989; 1991). Teachers are key agents in implementing these changes. Research indicates that teachers' beliefs about mathematics and its teaching influence their instructional decisions and practices (Thompson, 1992). However, many preservice and inservice teachers hold beliefs that conflict with the aims of the reform movement (Ball, 1990; Cooney, 1996; Gregg, 1995). Thus, one goal of teacher education is to help preservice and inservice teachers examine their existing beliefs as they experience alternative approaches to learning and teaching mathematics.

In this study, we examined two preservice secondary mathematics teachers' beliefs about learning and teaching mathematics. The participants were involved in several different activities that were designed to elicit the participants' beliefs. These activities included an interview in which the participants responded to a list of similes[1] about learning mathematics and being a mathematics teacher, a problem-solving session in which the participants were asked to solve routine and nonroutine calculus problems, and a final interview in which the participants were asked to reflect on their beliefs and actions. As we gathered information about the participants' beliefs and observed their problem-solving strategies, we were interested in the educational benefits of the research project for the participants themselves. One purpose of our study was to investigate the effectiveness of the activities in encouraging the participants to examine their existing beliefs and learning behaviors. This paper reports the results for one preservice secondary mathematics teacher who participated in our study.[2]

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[1] These similes were adapted from a survey from the RADIATE project (DUE 9254475), University of Georgia.

## Theoretical Perspective

Shaw and Jakubowski (1991) identified six cognitive requisites for the change of teachers' beliefs and practices. According to Shaw and Jakubowski, teachers must: (a) experience a perturbation, (b) make a commitment to change, (c) construct a vision of what the classroom could be, (d) project themselves into that vision, (e) decide to make a change within a given context, and (f) compare their practice with their vision through reflection. These requisites for change are also relevant to the professional development of preservice teachers. Preservice teachers often have a traditional image of mathematics teaching that was constructed from their experiences as students (Ball, 1990). Teacher educators may intervene to help preservice teachers examine their existing image of mathematics and its teaching. As a result, preservice teachers may begin to change their existing beliefs and behaviors. The six requisites identified by Shaw and Jakubowski provide a framework to help us identify and understand important aspects of teacher change and development.

## Methods

Jill was one of two preservice secondary mathematics teachers who agreed to participate in our study. Data were collected from an initial interview, a problem-solving session, and a post hoc interview. At the time of the initial interview, the participants were enrolled in their first mathematics education course, which promoted views of mathematics learning and teaching congruous with the NCTM *Standards*. In this interview, we asked each participant to consider a list of similes for learning mathematics and for being a mathematics teacher. Each participant was interviewed separately and asked to choose the similes that best described learning mathematics and being a mathematics teacher. The participants were then asked to discuss the reasons for their choices. These questions were "designed to gather information on beliefs about mathematics and its teaching" (Cooney, Shealy, & Arvold, in press).

Following the initial interview, we devised a problem-solving session to investigate the relationship between the participants' beliefs (as expressed by their responses to the similes) and their approaches to solving routine and nonroutine problems. For the problem-solving session, we chose two calculus problems. The first problem was similar to the types of problems seen by students in traditional calculus courses. The second problem did not have a clearly foreseeable right or wrong answer and was designed to encourage alternative methods of solution. We conducted the problem-solving session during the summer following the participants' completion of their first mathematics education course. During the problem-solving session, the participants worked together to solve the two problems. Immediately following this session, we conducted individual interviews in which participants were asked to reflect on their actions in the problem-solving session and to discuss how their actions were related to their beliefs (as expressed by their responses to the similes) about mathematics and its teaching.

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[2] A complete report of this study by Herbst, Mesa, and Gober is forthcoming. Other aspects are discussed in Mesa & Herbst (in press) and Herbst (in press).

## Results

Jill compared learning mathematics to working a jigsaw puzzle. In her words, learning mathematics is like working a jigsaw puzzle because "you don't know what's going to go where and [you] just kinda have to play with it and figure it out. A lot of math problems, you have to play with them to try to figure them out." In Jill's view, learning mathematics is an activeprocess, unlike watching a movie. She explains, "I'm always playing around with different stuff to see what I can come up with or what I can figure out on my own. But watching a movie, you're just sitting there. I mean you don't do anything."

As Jill considered the similes for a mathematics teacher, she faced a perturbation: "none of these match." However, Jill was committed to the task of finding an answer, and she defined her image of a good mathematics teacher by reinterpreting her past experiences. Jill's image of a good mathematics teacher stood in contrast to the characteristics of her own mathematics teachers. When Jill was first asked to think about what a mathematics teacher is like, she responded, "Okay, like you want me to think about what a mathematics teacher should be or like my experiences?" She compared her mathematics teachers to broadcasters: "They stood up there and told you what to do and left it at that." Jill went on to say that she did not see mathematics teachers as doctors or gardeners. However, after further consideration, these similes brought to mind some characteristics that Jill ascribed to a good mathematics teacher.

> A doctor to me is just somebody to, ooh, I can kinda see them as a doctor. . . . Because a doctor is always trying to help and a math teacher should be somebody that's trying to help you understand mathematics. . . . I don't see it as a gardener . . . A gardener. Maybe I can. I can, I can. Because a gardener is trying . . . to make plants grow, and be pretty, and blossom and all that. A mathematics teacher should be that, should be trying to nurture their students and help them understand mathematics

and grow and learn and live mathematics.

As Jill considered each of the other similes, she further defined her image of a good mathematics teachers: being open to different ideas (unlike a missionary), caring about students (unlike an entertainer), and sharing leadership roles with students (unlike a conductor or a coach). Thus, our discussion of the similes provided an opportunity for Jill to expound on her beliefs about mathematics, its learning, and its teaching.

Jill also faced a perturbation during the problem-solving session. In approaching the first problem (a routine calculus problem), Jill and Jack (the second participant in our study) decided to apply a standard algorithm. Jill lacked confidence in her ability to solve the problem in this way: "I cannot remember. . . . I don't remember any of this. I don't remember any math." Jill's lack of recall led her to try a different strategy on the second problem (a nonroutine problem involving the comparison of areas under curves). When Jack asked what she was doing, she replied, "Just playing." However, Jill's strategy eventually led to a solution to the problem and Jill's confidence increased. This new sense of confidence was revealed in statements such as, "I'm telling you Jack. I've got it figured out" and "My way worked!" Thus, Jill's initial perturbation and commitment to solving the problems led her to construct a different approach to the second problem. As Jill indicated in the final interview, she came to see the value of her own approach for the teaching of mathematics.

Following the problem-solving session, Jill reflected on what she had learned from her participation in the study. She had an opportunity to project herself into her vision of what a good mathematics teacher should be. Jill stressed that she needed to review her mathematics. However, Jill also recognized the value of her own "simpler" or "visual" approach to problem solving. Jill indicated that as a teacher, she would adapt to the different learning styles of her students by using different approaches to problem solving (e.g., the approaches that she and Jack used). She reaffirmed her belief that mathematics teachers should be open to different ideas. As Jill noted in the first interview, students "may come up with something totally different and it could be a good way of looking at it." This willingness to adapt to different learning styles and consider students' ideas reflects a desire to help and nurture students in their understanding of mathematics.

Our research study benefited Jill by providing opportunities for her to examine and expound on her beliefs about mathematics, its learning, and its teaching. The activities in which Jill was involved seemed to stimulate perturbations, foster her construction of an image of a good mathematics teacher, and provide opportunities for reflection that helped Jill to project herself into her image of a good mathematics teacher. As Shaw and Jakubowski suggest, these may be important factors in promoting teacher change and development.

## Conclusions

This report shows how one preservice teacher's participation in a research study provided opportunities for her to examine and expound on her beliefs about mathematics, its learning, and its teaching. Shaw and Jakubowski's framework helped us to identify the preservice teacher's cognitive levels as she responded to the similes, solved mathematical problems, and reflected on her beliefs and actions. By providing opportunities for preservice teachers to examine their beliefs and actions, we may create environments in which preservice teachers can experience the cognitive requisites identified by Shaw and Jakubowski (1991). In this way, we may stimulate changes in preservice teachers' beliefs about mathematics and its teaching.

## References

Ball, D. L. (1990). Breaking with experience in learning to teach mathematics: The role of a preservice methods course. *For the Learning of Mathematics, 10*(2), 10-16.

Cooney, T. J. (1996, July). *Conceptualizing the professional development of teachers*. Paper presented at the International Congress for Mathematical Education, Seville, Spain.

Cooney, T. J., Shealy, B. E., & Arvold, B. (in press). Conceptualizing belief structures of preservice secondary mathematics teachers. *Journal for Research in Mathematics Education.*

Gregg, J. (1995). The tensions and contradictions of the school mathematics tradition. *Journal for Research in Mathematics Education, 26*(5), 467-486.

Herbst, P. (in press). Effectiveness of a strategy as a sociomathematical norm. *Proceedings of the Twenty-first Conference of the International Group of Psychology of Mathematics Education*, Lahti, Finland.

Mesa, V., & Herbst, P. (in press). A problem solving session designed to explore the efficacy of similes of learning and teaching mathematics. *Proceedings of the Nineteenth Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Normal, Il.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: The Council.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: The Council.

Shaw, K. L., & Jakubowski, E. H. (1991). Teachers changing for changing times. *Focus on Learning Problems in Mathematics, 13*(4), 13-20.

Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127-146). New York: Macmillan.

# PRESERVICE MATHEMATICS TEACHERS' CONSTRUCTIONS OF GENDER EQUITY IN THE CLASSROOM

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The purpose of the study was to identify preservice teachers' perspectives on gender equity and to explore how they process the information they receive about gender issues. Data were collected through written surveys administered to approximately 250 preservice teachers enrolled in mathematics and science education (methods) courses at the university. Data were analyzed using Paine's (1990) categories of orientations to diversity: individual difference, categorical difference, contextual difference, and pedagogical difference. The study suggests that most preservice teachers have an individual difference view of gender equity in which they strive to treat all students the same to avoid discrimination. Some students hold a categorical view of gender equity in which they strive to overcome stereotypes about boys and girls. A small number of students actively denied that gender issues have any relevance to education.

> It is unnecessary, year after year, to graduate new classroom teachers who, because they don't know any better, unintentionally diminish the educational, and therefore career, and therefore economic prospects of the female half of the population and thus of the nation. We can and must do better. (Campbell & Sanders, 1997)

Integrating Gender Equity and Reform (In GEAR[1]) is a three year collaborative project being conducted by Clark-Atlanta University, the Georgia Institute of Technology, Georgia Southern University, Georgia State University, and the University of Georgia. Each institution has a team of mathematics educators, science educators, mathematicians, scientists, and others working on the project on its campus. The American Association of University Women of Georgia and the Georgia Initiative in Mathematics and Science (a statewide systemic initiative, also funded by the National Science Foundation) are also collaborating on the project. The purpose of the project is to change the ways in which preservice elementary, middle, and high school teachers learn to teach science and mathematics. The project has two main objectives: To initiate and implement the redesign of teacher preparation programs, including instruction in science, engineering, and mathematics courses, so that teachers entering K-12 classrooms are able to address issues that discourage girls and women from participating in scientific and technological fields; and to provide professional development opportunities for faculty and teaching assistants that will equip them with positive support and intervention strategies.

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[1] This material is based upon work supported by the National Science Foundation under Grant No. HRD-9453106. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the National Science Foundation.

To achieve the project goals, the project has four strands: an institutional self-evaluation; professional development of faculty and teaching assistants; a toolkit[2] of materials for teacher preparation courses; and a framework for teacher education. The collaboration was designed so that each educational institution in the partnership would take a leadership role for one strand, but all campuses would implement each strand.

The study that is reported in this paper was conducted as part of the institutional self-evaluation. The purpose of the study was to identify the perspectives about gender equity that preservice teachers bring with them to their methods classes. We were also interested in knowing how preservice teachers process the information they receive about gender issues. Specifically, we were interested in knowing what teaching strategies these teachers thought they might employ in their own classrooms to create a gender equitable learning environment. Knowing how preservice teachers think about gender issues as they enter their teacher education program and as they are exposed to information about gender issues can provide the mathematics teacher education community with information about what aspects of gender equity might be addressed and how they might be addressed in ways that are most meaningful to teacher education students.

## Theoretical Framework

Data were analyzed using Paine's (1990) categories of orientations to diversity. Paine's categories were developed to analyze prospective teachers' views of cultural diversity, but the categories seem relevant for the specific diversity issue of gender as well. Paine's framework consists of four layers of meaning for diversity: individual difference, categorical difference, contextual difference, and pedagogical difference. From an individual difference perspective, people are seen as different in all dimensions. Preservice teachers holding this orientation toward diversity try to respond to students on an individual basis. The source and solution of a problem will depend on the individuals concerned.

In a categorical view of diversity, preservice teachers associate specific characteristics and patterns of difference with people in various categories. These categories include social class, race, and gender. However, the social construction of these categories is not examined, so the categories themselves remain unchallenged. For example, in the category of gender, this means that efforts to address gender differences focus on removing barriers to females' participation or on changing females. In this view, the problem is with the females and the ways in which they have been socialized.

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[2] The toolkit can be found on the world wide web at http://www.coe.uga.edu/ingear.

The third orientation, contextual difference, connects patterns of differences to a social situation. For example, gender differences are not fixed but constructed through social interaction. This approach takes into account the causes of difference, unlike the individual or categorical orientation. Teachers with a pedagogical orientation understand that differences have implications for teaching and learning. An understanding of differences is combined with knowledge of equitable teaching strategies.

Paine (1990) used these four categories to analyze preservice teachers' orientations toward diversity. In a survey of elementary and secondary preservice teachers, Paine found that preservice teachers relied heavily on an individual difference orientation to teaching. They indicated the importance of fairness and equality for all students, but rejected certain differences (e.g., gender) as having important implications for teaching. When asked about specific teaching practices that would address diversity in the classroom, these preservice teachers generally responded with vague or confusing answers.

## Data Collection

Written surveys were administered to approximately 250 preservice teachers enrolled in mathematics and science education (methods) courses at the University of Georgia. Approximately half of the preservice teachers surveyed were early childhood education majors, one-fourth were middle school education majors with a primary or secondary concentration in mathematics or science, and one-fourth were secondary mathematics majors. The preservice teachers were at various phases of their teacher education program, ranging from their second quarter of professional education coursework to a post-student teaching seminar.

Data were analyzed using a coding scheme developed by Campbell (1995) as part of the Teacher Education Equity Project. The coding scheme involves reading participants' responses and rating whether their response reflects lack of awareness of gender issues, some awareness of gender issues, or action with respect to gender issues. Those responses coded as "awareness" are further coded as to whether they show neutral or negative awareness. Responses coded as "action" are further coded as to whether they show non-specific action, isolated action, or integrated action. These codings were then used to identify which of Paine's categories of orientation toward diversity best described the participant.

## Results

The data suggest that most preservice teachers have an individual or categorical difference perspective on gender. Approximately 10% of the students surveyed could not be categorized according to Paine's scheme. These students actively denied the existence of gender equity issues in the classroom. They made statements such as, "There really is no need to be concerned with gender, just the quality of the instructor and their teaching practices." or "Personally, I believe it is an issue that is irrelevant to successful education." or "I do not have a problem with feeling suppressed or discriminated against in my science classes. I do not see a problem [because I] have not experienced this in my classes." Several other students noted that they had not ever considered the issue of gender equity, and therefore they did not complete the survey.

Approximately two-thirds of the students' responses were classified as reflecting an individual difference perspective. Students with an individual difference perspective noted that they would treat all students the same in order to avoid discrimination. They used words such as "equal," "the same," or "fair" to describe a gender equitable teacher. These students seemed to believe that by ignoring gender their classrooms would be equitable. Representative student comments include "[A gender equitable teacher] doesn't prefer one gender over the other." or "[A gender equitable teacher] doesn't call on one gender more than the other." or "Students in a classroom are treated equally no matter what gender they are." When asked what strategies they would employ in their own classrooms to ensure that their students were receiving gender equitable instruction, these students tended to suggest strategies such as alternating calling on boys and girls and using tally systems to ensure that both genders were being called on equally.

Approximately one-fourth of the preservice teachers' responses reflected a categorical orientation toward gender. Students with a categorical difference perspective saw gender as defining how students perform and react to mathematics instruction. One student said that a gender equitable teacher is one who can "discuss the differences between girls and boys and recognize them." Another student said that a gender equitable teacher is one who "informs children that they can do anything that the opposite gender can do." Yet another student said that a gender equitable teacher "acknowledges that females are capable of achievements in math and science" and "females are given an appropriate amount of time/attention in the classroom." When asked what they would do in their own classrooms to ensure that their students were receiving gender equitable instruction, these students tended to suggest strategies that debunked gender stereotypes. For example, students suggested using literature and guest speakers to portray women in typically masculine roles. One student noted the importance of avoiding word problems about baseball or dolls.

Students enrolled in a section of an early childhood mathematics methods class where the instructor had devoted one class period to discussion of gender issues account for nearly 50% of the students classified as holding a categorical view. The remaining 50% of students who held a categorical view came from two other classes in which the instructors had explicitly addressed gender equity in class, although to a lesser extent than the first instructor.

## Implications for Teacher Education

The data suggest that most of the preservice teachers in this study are likely to unintentionally diminish the educational opportunities of female students because they do not know any better, as Campbell and Sanders (1997) warn. Preservice teachers need opportunities to examine their views of gender equity and to read literature that poses other views. Further, because most of the participants in this study were unaware of specific strategies they might employ in their own classrooms, preservice teachers also need to learn about and try various instructional strategies that have been shown to produce an equitable classroom environment.

Indeed, most of the preservice teachers in this study indicated that their professional education classes had seldom or never addressed gender equity issues. It is important for teacher education faculty to address gender equity issues (as well as multicultural issues) in a systematic manner. It is interesting to note that even the students who had some exposure to instruction about gender equity reflected individual and categorical orientations toward gender equity. Obviously, a one or two hour lecture on the topic is not sufficient.

As with most topics in teacher education, reading about a topic and taking lecture notes is insufficient for students to operationalize the ideas in the classroom. Therefore, an important component of teacher education experiences should be the analysis of and reflection upon classroom practice–their own classroom practice and that of peers and mentor teachers. By reflecting on classroom practices, preservice teachers may be able to identify gender equity issues and strategies that are most salient in their own teaching.

## References

Campbell, P. B. (1995). *TEEP participant equity statements: A summary of the results.* Unpublished manuscript.

Campbell, P. B., & Sanders, J. (1997). Uninformed but interested: Findings of a national survey on gender equity in pre-service teacher education. *Journal of Teacher Education, 48*(1), 69-75.

Paine, L. (1990). *Orientation toward diversity: What do prospective teachers bring?* (Research Report 89-9). East Lansing, MI: National Center for Research on Teacher Education. (ERIC Document Reproduction Service No. ED 320 903)

# THE PERCEPTIONS OF PRESERVICE ELEMENTARY TEACHERS ABOUT THE INTEGRATION OF MATHEMATICS AND READING

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There is a profusion of discussion and literature about the use of mathematics and language arts together in the elementary classroom. This research examines the perceptions of elementary preservice teachers about the desirability and ease of integrating mathematics and reading. In this pilot study, 54 subjects gave responses to 12 Likert-type statements. These subjects were in three methods classes: mathematics methods, reading methods, and integrated reading and mathematics methods. The responses of the three classes on pre and post surveys are given. There was a significant difference in perceptions on one of the questions, concerning the ease of actually integrating the two subjects in the classroom. In addition, on the post surveys there were significant differences between the three classes on the post survey concerning confidence in ability to teach integratively and concerning the relationship of problem solving abilities and reading.

## Purpose

Attempts to integrate mathematics and reading instruction can be seen in curriculum materials and textbooks suggesting how reading instruction can contribute to achievement in mathematics (e.g., Ferguson & Fairburn, 1985; Kresse, 1984; Shell, 1982; Singer & Donlan, 1980). However, most of the efforts to integrate mathematics and reading have not yet succeeded in fully exploiting the potential for an educational agenda aimed at using reading as a way of promoting mathematical reasoning skills among public school students and teachers. Further, some might argue that teacher education has, for the most part, neglected curriculum and instructional integration. The purpose of this study was to investigate the perceptions of preservice elementary teachers about integrating the teaching of reading and mathematics. Of particular interest was the difference in perceptions about integrating the teaching of reading and mathematics between those preservice teachers enrolled in a course that stressed integrating instruction and those enrolled in courses that did not stress integrating such instruction. The perceptions that these prospective teachers have will determine whether or not they will use integration techniques in their classrooms.

## Theoretical Framework

There is a call for a new synthesis of mathematics and reading. That is, there are efforts to recognize reading is a mode of learning that can be utilized to give readers connections and strategies in the mathematics curriculum (Siegel, Borasi, & Smith, 1989). In recent years, there has been a significant shift in the ways that the disciplines of mathematics and reading have been conceived. In both fields, the shift from transmission to transactional or constructivist views of learning has been well documented. Reading and language arts educators have developed curricula which use activities that promote increased learning through reflection and problem solving. Examples of such activities include using children's literature to learn mathematics (e.g., Whittin & Wilde, 1992), exploring writing as a way to learn content areas and to promote critical thinking skills (e.g., King, 1982), and authoring curricula which are based on the idea that meaningful learning occurs when students use multiple communication systems in context (e.g., Harste, Woodward, & Burke, 1984; Rowe & Harste, 1986).

Reading plays a role analogous to the one which writing has recently begun to play in mathematics instruction. Arguing that writing is a mode of thinking and not simply a matter of transcribing thoughts into words, mathematics educators have proposed that writing experiences can support mathematics learning (King, 1982; Stempien & Borasi, 1985).

## Methods

The participants in this pilot study were 54 preservice elementary teachers at Oklahoma State University. They were enrolled in three sections of curriculum and instructional methods classes: a math methods section (n = 21), a reading methods section (n = 13), and a special reading methods section where subjects received training in teaching reading and math integratively (n = 20). Each subject tutored an elementary school-age child in both math and reading in partial fulfillment of the course requirements. Subjects in the math-reading section used math-reading integration strategies learned for their tutoring assignment. Subjects in the other two classes followed a regular math or reading methods curriculum where they learned to teach math or reading independently.

All of the preservice teachers completed a survey, which was prepared specifically for this exploratory study, at the beginning and at the end of the Fall semester, 1996. The survey was designed to determine perceptions about integrating the teaching of math and reading at the elementary/middle school level. The survey instrument consisted of statements that solicited the subjects' responses on a Likert-type scale, ranging from 1 (Strongly Disagree) to 5 (Strongly Agree). Statements about the effectiveness of integrating math and reading in the classroom, using math to teach reading, and using reading to teach math were included.

To ensure that the statements included in the survey were appropriate and valid, a number of steps were taken. First, available research studies, which tap teachers attitudes and perceptions about reading and math in general, were examine. Second, we took into consideration extensive feedback solicited from discussions with colleagues and students concerning the clarity and appropriateness of the statements. Third, we piloted an earlier version of the survey with a small sample of preservice and inservice teachers. All these steps resulted in refinements to the survey as a whole. The survey instrument had an internal consistency reliability of .89 for the statements suggesting that the scales were measuring the same construct. The survey questions, along with the results obtained, may be found in Table 1.

## Results And Discussion

The data obtained were analyzed using repeated measures ANOVAs with group (Reading, Math, Reading-Math) as independent variables, and the time factor as a repeated measures variable (each subject completed the survey twice: at the beginning of semester and at the end). We were interested in finding out whether a group of preservice teachers' perceptions about several aspects of the integration of language arts and mathematics changed as a result of completing a methods course where they learned to teach math and reading integratively.

The results, presented in Table 1 below, revealed two important findings. First, when the subjects were asked to report their feelings toward the integration of the language arts and mathematics, it was found that there was a relatively high degree of agreement among the subjects in all three groups—at the beginning and at the end of the semester—about the interdependence of the language arts and mathematics ((= .86) as indicated by the mean ratings reported for each of the dependent variables. The subjects' agreement seems to indicate their feelings that reading, writing and math naturally support each other, and that they should be learned and taught together.

Second, while we anticipated that the group which underwent training in the integration of reading and math to indicate higher ratings towards the integration of the two subjects than the other two groups, we found no statistically significant differences among any of the three groups with respect to 11 of the 12 dependent variables. We attribute the lack of significance in this case, at least in part, to a ceiling effect due to the fact that nearly all of the subjects, regardless of group membership, had relatively high mean ratings of their feelings towards the integration of reading and mathematics. The only significant difference noted was between the reading/math group and the reading group, [F (2, 51) = 6.17, p =.004] with respect to their feelings about whether or not language arts and mathematics can be easily combined in the classroom (Statement # 7).

**Table 1** *Means, Standard Deviations, and F-Tests*

| DEPENDENT VARIABLES | GROUPS | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | MATH | | | | READING/MATH | | | | READING | | | | | |
| | PRE | | POST | | PRE | | POST | | PRE | | POST | | F | |
| | M | SD | M | SD | M | SD | M | SD | M | SD | M | SD | Value | P>F |
| 1. Improving reading skills can lead to better mathematical understanding. | 4.57 | 0.68 | 4.43 | 0.68 | 4.45 | 0.60 | 4.70 | 0.47 | 4.15 | 0.90 | 4.46 | 0.78 | 0.92 | 0.40 |
| 2. Children often do not understand math concepts because they do not understand the language used to communicate those concepts. | 4.67 | 0.58 | 4.62 | 0.50 | 4.75 | 0.44 | 4.80 | 0.41 | 4.62 | 0.51 | 4.85 | 0.38 | 0.58 | 0.56 |
| 3. Math and reading cannot be effectively taught together. | 1.57 | 0.98 | 1.67 | 1.10 | 1.40 | 0.50 | 2.05 | 1.23 | 1.54 | 0.88 | 1.15 | 0.38 | 1.14 | 0.33 |
| 4. Children's literature helps give a meaningful context for math ideas. | 4.48 | 0.75 | 4.52 | 0.68 | 4.45 | 0.51 | 4.60 | 0.68 | 4.38 | 0.77 | 4.54 | 0.78 | 0.05 | 0.96 |
| 5. Writing is not useful for learning math. | 1.24 | 0.44 | 1.29 | 0.46 | 1.70 | 1.13 | 1.45 | 1.00 | 1.07 | 0.28 | 1.46 | 1.13 | 1.53 | 0.23 |
| 6. Most children like reading more than they do math. | 3.57 | 0.81 | 3.86 | 0.79 | 3.55 | 0.95 | 3.40 | 1.19 | 3.54 | 0.97 | 3.48 | 1.05 | 0.47 | 0.63 |
| 7. Language arts and math can easily be combined in the classroom. | 4.33 | 0.66 | 4.43 | 0.68 | 3.85 | 0.99 | 3.55 | 1.05 | 4.08 | 0.64 | 4.69 | 0.48 | 6.17 | 0.004* |
| 8. Students' math achievement is not related to reading achievement. | 2.29 | 1.01 | 2.14 | 0.96 | 1.65 | 0.67 | 2.10 | 1.12 | 2.08 | 0.76 | 1.85 | 0.99 | 1.12 | 0.33 |
| 9. Students who write their own math problems become better problem solvers. | 4.23 | 0.77 | 4.24 | 0.70 | 4.25 | 0.91 | 4.80 | 0.41 | 4.15 | 1.14 | 4.31 | 0.75 | 1.29 | 0.28 |
| 10. Teachers can use children's literature to teach math. | 4.90 | 0.30 | 4.67 | 0.48 | 4.55 | 0.60 | 4.55 | 0.69 | 4.62 | 0.65 | 4.77 | 0.44 | 1.62 | 0.21 |
| 11. Writing in a math class does not help with math problem solving. | 1.43 | 0.51 | 1.29 | 0.46 | 1.30 | 0.57 | 1.20 | 0.41 | 1.38 | 0.51 | 1.46 | 0.88 | 0.64 | 0.53 |
| 12.Teachers can use math to teach children's literature. | 3.76 | 0.94 | 3.71 | 1.15 | 3.50 | 0.76 | 3.20 | 1.11 | 3.92 | 0.76 | 4.08 | 1.26 | 2.49 | 0.09 |

* Statistically significant

Furthermore, it was found that the subjects in the reading/math methods course felt somewhat less confident about teaching math and reading together at the end of the semester (M = 3.55; SD = 1.05) than did the subjects who learned how to teach reading only (M = 4.69; SD = .48). This finding, while intriguing in light of the fact that the subjects in the math/reading group were "trained" to teach math and reading, can be explained by the fact that these preservice teachers realized that the more they knew about why and how to integrate reading and math as well as about the challenges involved in the process-information gained during the course of an entire semester through readings, class discussions, tutoring experiences, and interviews with practicing teachers—the less confident they felt about their personal abilities to teach language arts and mathematics together. This particular issue has raised several related questions regarding the potential for as well as the difficulties inherent in any attempt to integrate curricula. These questions are currently being addressed in a follow-up study.

Finally, an analysis of the group responses for the post survey only showed a significant difference between the subjects in the math group and the math/reading group with respect to the relationship between writing math problems and problem solving [(F(2,51) = 4.70; p = .01]. Specifically, in their response to statement #7, the subjects' the reading/math group felt more confident (M = 4.80; SD = .41) about the usefulness of teaching children about solving as well as writing or creating math problems than did the subjects in the math group (M = 4.23; SD .70). This difference was evident in the subjects' lesson plans as well as the reflective journals they kept while tutoring an elementary school child.

The findings in this pilot study have important implications for curriculum leaders, language arts and mathematics researchers and practitioners. First, the results do lend some support to those researchers who have advocated the teaching and learning of reading and mathematics integratively. These results suggest that teacher educators ought to begin to design programs in support of interdisciplinary reading and mathematics curricula.

Second, it is clear from the literature available in this relatively "new" area of research that our goal as language arts and mathematics teachers should be to increase our understanding of the instructional, assessment, administrative, and psychological aspects of integrating curricula particularly with respect to reading and mathematics. As classroom teachers begin to explore the potential for using reading and writing as means of enhancing mathematics learning and problem solving, the value of the integration of reading and math will become more and more evident. Similarly, as educators recognize the importance of enhancing children's learning and achievement, the need for designing supportive curricula becomes more obvious.

The findings of this study are encouraging, especially to those teachers who are concerned about their students' reading, writing and math skills. Integrating reading and writing with math may be used as a way of motivating students to read and write and have enjoy doing it. However, since the study was conducted with a small intact group of preservice teachers, its generalizability is limited. More in-depth research with a higher number of preservice and inservice teachers is strongly suggested.

## References

Ferguson, A. & Fairburn, J. (1985). Language experience for problem solving in mathematics. *Reading Teacher, 38*(6), 504-507.

Harste, J. C., Woodward, V. A., & Burke, C. L. (1984). *Language stories and literacy lessons.* Portsmouth, NH: Heinemann.

Kresse, E. (1984). Using reading as a thinking process to solve math story problems. *Journal of Reading, 27*(7), 598-601.

Rowe, D., & Harste, J. (1986). Reading and writing in a system of knowing. In M. Sampson, (Ed.), *The pursuit of literacy* (pp. 126-144).

Shell, V. (1982). Learning partners: Reading and mathematics. *Reading Teacher, 35*(5), 544-548.

Siegel, M., Borasi, R., & Smith, C. (1989). A critical review of reading in mathematics instruction: The need for a new synthesis. *NCR Proceedings, Cognitive and Social Prospectives*, 269-277.

Singer, H., & Donlan, D. (1980). *Reading and learning from text.* Boston: Little, Brown, and Company.

Stempien, M., & Borasi, R. (1985). Students' writing in mathematics: Some ideas and experiences. *For the Learning of Mathematics*, 14-17.

Whitin, D. J., & Wilde, S. (1992). *Read any good math lately?* Portsmouth, NH: Heinemann.

# THE IMPACT OF MATH APATHY STUDENTS ON ONE HIGH SCHOOL TEACHER

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Mrs. Wells, an Algebra I teacher, struggles to instill a desire of learning in all her students while at the same time dealing with students who do not care. On the one hand, she "hates" having students with math apathy in her class, because they do not want to learn and many times they are a disruption to students who do want to learn. On the other hand, Mrs. Wells sincerely takes a personal interest in each student and wants every student to be successful. A case study was done to better understand Mrs. Wells' struggles. The results show that the rigid mathematics curriculum and her own attitudes and beliefs about learning and teaching shaped what she did in the classroom. The most surprising of the findings for her was the realization that her methods and attitudes directly contributed to the students' apathy. As a result of this realization, she made significant changes in her teaching.

One of the greatest challenges mathematics teachers face is to create a culture for learning that is both mathematically challenging yet inviting to all students. The *Professional Standards for Teaching Mathematics* (NCTM, 1991) advocates the proper use of four main areas that help make up the classroom culture: mathematical tasks, discourse, environment, and analysis. How does a teacher implement these when faced with students who are apathetic toward mathematics? How do students with math apathy affect the classroom culture? How do they impact the teacher's beliefs and teaching methods? How do the textbook and its underlying philosophy affect the culture, the students' beliefs and attitudes, and the teacher's beliefs, attitudes and actions? During the 1996-1997 school year, we set out to better understand these questions in Mrs. Wells'[1] Algebra I class.

## Theoretical Perspective

Our views of knowing, culture, beliefs, and teacher change partly shape how we view and analyze the data. We believe that knowledge is constructed individually (von Glasersfeld, 1987, 1989) and shaped largely by prior experiences and the culture in which it is learned (Davis, Shaw, & McCarty, 1993). Rossman, Corbett, and Firestone (1988) explicate three cultural change processes, one of which seems relevant to this study. The process, known as "additive," refers to beliefs that are modified quite suddenly and spread to entire belief systems. To understand this cultural change, it was important to understand Mrs. Wells' beliefs at the beginning of the study and as she modified them throughout the year. We utilized our understanding that beliefs are often clustered (Green, 1971), sometimes being interrelated with other clusters and sometimes being isolated from others. We also realized that these clusters of beliefs can be very contextually and culturally dependent (Ashley, 1996).

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[1] Mrs. Wells is the second author of this paper.

## Methods

We conducted the study in Mrs. Wells' natural environment, as learners, not ones that know all the answers, with the understanding that human instruments are involved and therefore qualitative methods are optimal, and by understanding the importance of one's own tacit knowledge to make inferences (Guba & Lincoln, 1989). The method used to investigate Mrs. Wells' struggles was the case study. Data collection tools were: interviews, journals, and observations. The interviews (10-12) took place in Mrs. Wells' classroom, in the office of the first author, or on e-mail. A journal (22 entries) was kept by Mrs. Wells. She reflected on what was occurring in her classroom as it related to her own teaching and the learning of her students. Fieldnotes were taken immediately following observations (5) made during the class. The data sources were used collectively to better understand the struggles Mrs. Wells was having.

Mrs. Wells is 40 years old and has taught high school mathematics for three years. She completed her master's degree in mathematics education in the spring of 1997. She was selected for this project because of her desire to learn more about the teaching methods and their impact on math apathy students.

## Results

*You don't care, I don't care.* Mrs. Wells writes in her journal that God has given her "an incredible love for students and teaching" and that she is blessed to have an opportunity to practice what she loves to do. Her love for teaching is evident during the observations of her in the classroom. She is energetic. She presents her subject matter in a clear and direct manner. She talks to each student several times in class and she demonstrates genuine care for each student. Even with all of these qualities, she has become discouraged with several of her 18 students in Algebra I. "I don't want to give up hope on these students, but I've started to believe if they don't care, why should I? All I really want is for the student to care. They don't have to be brilliant, but I want them to at least care." She addressed this concern in her journal,

> I am very tired of those three or four students whose seemingly life's purpose is to see what they can get away with. These "3 or 4" students affect most everything I do or don't do for that matter. I don't have an

> answer, in fact, I feel completely drained of ideas, and that is the most frustrating part about it. (Dec 11, 1996)

The frustration was heightened by her belief that all she does should be of excellent quality and that students should have this same goal of excellence.

*Striving for excellence is sometimes my weakness.* Mrs. Wells, by her very nature, strives for excellence in all she does. She believes it is a "calling." She elaborates, "I know that this attitude in me can sometimes become one of my weaknesses. In other words, since I expect a lot of myself, I expect my students to be of 'like mind.'" It was evident that Mrs. Wells has difficulty having empathy for her math apathy students. She does not understand students who are not striving for excellence. One day, Mrs. Wells told her students who needed to make up a test that they must come after school next Tuesday to take it or they would get a zero. She reminded them several times including the day of the test. Tuesday afternoon, she posts these reflections, "It's now 3:30 and I have not had any of my students come to take their make-up test. I relinquish all responsibility and will write down a 0 in the grade book with no pangs of guilt." She goes on to say that it saddens her to realize the irresponsibility of these 16 year old students. As a result of conducting this research, she became aware of her apathy towards the apathetic students. She wanted to change, but felt confined by the curriculum.

*Do 30 problems every night/Understanding will come.* The mathematics textbooks heavily influenced Mrs. Wells' teaching and her view of how students learned. Some of the publisher's comments about using the textbooks were,

> Don't try to teach students to think by playing Socrates and asking questions so that they will invent the math they need to know. You did not learn math this way and neither will they. In-depth discussions and cute questions should be considered only after a long period of time. . . . The teacher's primary job is finding a way to get each student to work all the problems.

Even though philosophically she did not agree with this, she felt that to be successful in using the textbooks, she must adhere to the publisher's recommendations. After attending a 2-day seminar sponsored by the publishers last summer, Mrs. Wells was told to be very rigid in what she did as a teacher each day. The daily routine was to: check homework (5 minutes), teach the lesson (10 minutes), do homework in class (40 minutes).

Mrs. Wells believes that if students do 30 problems a night, they will do well in algebra and subsequent courses. After the 2-day summer seminar, she was convinced that her past problems "must have been because I was not adhering strictly to that particular philosophy of teaching math." In trying to understand why, Mrs. Wells responded, "That is what you are supposed to do when you use these books." She validated this by talking to a past Algebra I student who had the same opinion.

However, she began to question the approach of covering one lesson per day when teaching the topic of the rectangular coordinate system. She wrote in her journal how she thought it would be best to stay on a topic for several days until all her students understand, especially those "less gifted (and motivated)" students. But she knew to be "successful" meant teaching a new lesson every day no matter how different the topic. The rigidity of the textbook became a recurring struggle.

*Teacher as Motivator.* Mrs. Wells loves to teach and loves being around young people, but she feels inadequate to deal with those who are apathetic. She tried to motivate them one day by talking about her feelings. Summarizing this talk, she said she could no longer continue teaching this way and feeling the way she does. She told the class she was going to change the way she taught the class. She admitted to them that the main reason she did not enjoy teaching Algebra I was because she felt more like a baby-sitter than their teacher.

### Transition

Mrs. Wells realized that the textbook reinforced her role as baby-sitter. She noticed that the rigid approach prescribed by the textbook publishers took away from her professionalism and her high calling of being a teacher. Throughout the semester she had wrestled much with her style of teaching and began making changes. In December 1996, Mrs. Wells realized her apathy and negative attitudes toward her math apathy students. She became more positive and began rewarding her students. During January 1997, she began to take a personal interest in those students who were often overlooked. On January 27, Mrs. Wells rearranged her room and partnered the students. She changed how she taught the lessons by extending the time spent teaching; she no longer followed the rigid schedule. On March 24, Mrs. Wells no longer strictly followed the textbook. She spent an entire week on graphing. In April Mrs. Wells chose the topics which she thought students needed to learn. She selected 10-15 problems from specific problem sets in the book for her students to complete for homework. Near the end of the study, she summarized her changes,

> I cannot and will not teach the [publisher's] way anymore. I did not enjoy it; in fact, I was miserable teaching that way which affected my classroom negatively. I do not begin to claim to know what the right way is; I just know the [publisher's] way is not it. I also no longer feel torn or hopeless as to what I should do as a teacher of algebra. In fact, I am excited about the challenges that lie ahead, no matter what text-

book we choose. For me, I now know that my success in the classroom is not found in trying to imitate others' ways, but in being myself and confident in my own style of teaching.

There was also a transition in her students. Initially, the students were frustrated because they were unable to really understand what they were learning which led to difficulties completing the 30 problems/day. Due to Mrs. Wells' changes, her students began to talk about their algebraic learning and understanding to Mrs. Wells and with one another. As a result, of her 18 students, 8 made D's or F's for the first semester and very few made As, but during second semester, half of her students had As for both the 3rd and 4th nine weeks with only one failing.

## Conclusions

Mrs. Wells was greatly impacted by the textbook and its philosophy, the students' apathy, and her apathy toward her students' apathy. Upon better understanding how these factors impacted her beliefs and actions, Mrs. Wells was able to make empowering decisions to reduce the rigidity of the curriculum, reduce her apathy toward her apathetic students, and raise the importance of the teacher in establishing a conducive and positive culture for learning.

Mrs. Wells still struggles with appropriate ways in which to deal with math apathy students. In fact, she probably always will. But she no longer struggles with her own apathy towards those students. She has relearned a valuable lesson—"real change always begins within oneself!" She realizes that students are not like her and may never have a high commitment to learn. She no longer views math apathy students in the same light: "If you don't care, I don't care." Instead, she has realized that many students need to be invited to learn. She seeks now to make her classroom and algebra more inviting.

## References

Ashley, D. (1996). *How different cultures impacted one teacher's beliefs in teaching mathematics.* Unpublished doctoral dissertation, Florida State University, Tallahassee.

Davis, N. T., Shaw, K. L., & McCarty, B. J. (1993). Creating cultures for change in mathematics and science teaching. In P. Rubba, L. Campbell, & T. Dana (Eds.), *Excellence in science teacher education.* Columbus, OH: ERIC/CSMEE.

Green, T. (1971). Teaching and the formation of beliefs. *The activities of teaching* (pp. 41-63). New York: McGraw-Hill.

Guba, E., & Lincoln, Y. (1989). *Fourth generation evaluation.* Newbury Park, CA: Sage Publications.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics.* Reston, VA: The Council.

Rossman, G. B., Corbett, H. D., & Firestone, W. A. (1988). *Change and effectiveness in schools: A cultural perspective.* Albany, NY: State University of New York Press.

von Glasersfeld, E. (1987). *Learning as a constructive activity.* Seaside, CA: The Systems Inquiry Series, Intersystems Publications.

von Glasersfeld, E. (1989). Cognition, construction of knowledge, and teaching. *Synthese, 80*(1), 121-1.

# DISTRICT-WIDE REFLECTIVE TEACHING IN MATHEMATICS: FROM CHANGING THE STORY TO STORING THE CHANGE

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The focus of this report is mathematics teaching in the context of implementing the Reflective Teaching Model (RTM), where pairs of teachers conducted systematic monthly planning for 0 to 4 years of at least one mathematics lesson with each other, observed each other teaching, and then followed up with a non-evaluative debriefing. The theoretical framework of the RTM is based on constructivism and metacognition, and a conceptual framework of the eight Levels of Use (LoU) of an innovation was used to study teacher luminal knowledge, which exists between the time teachers start letting go of previous knowledge and practice and before they reconstruct their knowledge and practice. Storytelling was used to describe knowledge and practice. Searches, merges, and reports assisted by NUD.IST software found patterns across groups of teachers according to the extent of time they have been working toward change and their position in the Levels of Use of the RTM.

This study is a survey of teacher knowledge in a district-wide elementary mathematics and science education project (CAMP) designed to facilitate teacher change in response to the national mathematics (NCTM, 1989, 1991) and science standards (NRC, 1996; Project 2061, AAAS, 1993). During the time frame of CAMP, from 1992 to 1996, a progressive teacher enhancement program implemented a process of restructuring teaching. The process, which is embodied in the Reflective Teaching Model (Hart, Schultz, Najee-ullah, & Nash, 1992; Schultz, Hart, Najee-ullah, Nash, & Jones, 1993; Keys & Golley, 1996; Wagner, 1994; Thomas, 1993), is continuing in this district beyond the funding period. For each new cohort of teachers (in combination with the previous year's cohort and teacher leaders), implementation occurs after a theory-based inservice on pedagogy and its association with national standards. The RTM process includes monthly planning of at least one mathematics or science lesson with a "reflective" partner who then observes the live or videotaped lesson and follows up with a non-evaluative debriefing.

## Objective

The objective of this study was to examine the knowledge about teaching mathematics acquired by elementary school teachers at the four-year mark of their district's participation in CAMP. The perspective of this research is the impact of the RTM on teachers in their mathematics teaching practice. We asked

*"What do teachers know?" and "How is teacher knowledge shaped by the professional knowledge context in which teachers work?"* Fenstermacher (1994) and Clandinin and Connelly (1996). The context of this study is the shifting landscape of professional knowledge in mathematics education, where the RTM was the teacher-driven means of changing that landscape.

## Framework

The theoretical framework is built on constructivism (Bauersfeld, 1992; Confrey, 1986, 1992; Schifter & Fosnot, 1993; von Glasersfeld, 1984, 1990), metacognition (Flavell, 1979; Schoenfeld, 1987, 1992), and negotiation of sociomathematical norms (Yackel & Cobb, 1996). The paradigm of mathematics instruction that is supported by this theoretical framework presents a great challenge which is aptly put by Schifter and Fosnot (1993) when they wrote that no matter how lucidly and patiently teachers explain *to* their students, they cannot understand *for* them (p.9).

The LoU, or Levels of Use of an innovation (Loucks, 1983) was the framework to study teacher's luminal knowledge. These levels are O: Non-Use of innovation, I: Orientation to innovation, II: Preparation to use, III: Mechanical use, IV-A: Routine use, IV-B: Refinement, V: Integration with others, and VI: Renewal toward more effective alternatives.

## Mode of Inquiry

We used story telling (Feldman, 1990; Clandinin & Connelly, 1996; Isenberg, 1995) to examine luminality (Feldman, 1990), which in this study means teacher knowledge after teachers start letting go of previous thinking and practice but before they've reconstructed it. It is the "betwixt and between position which develops when previous structural arrangements have terminated, but new ones have not yet been established" (pp. 809-810). Teachers' stories were their perceptions of the changing landscape of their world of teaching; our analysis of the stories found patterns across groups of teachers according to the extent of time they have been working toward change and their position in the Levels of Use of the RTM.

## Data Source

Stories were collected from 12 pairs of teachers in 12 elementary schools in a metro-Atlanta school district. At the request of the school district, all participating schools and teachers have been coded to maintain anonymity. All demographics, stories, and follow-up questions were recorded on one audiotape per teacher. Training for the data collection process was conducted by the Instructional Lead Teacher from each school, who previously was trained by us. All Instructional Lead Teachers have been facilitators of the RTM and have a unique relationship with those teachers in the context of the RTM which began first in mathematics, then science, and now in the other disciplines. Two teachers per each of the 12 elementary schools were selected by the district's Elementary Mathematics Coordinator to provide demographics, stories, and responses to follow-up questions on their story. The Elementary Mathematics Coordinator was instrumental in district-wide implementation of the RTM which began over five years ago at the time of this writing. The Coordinator selected teachers in order to have a distribution of teachers according to the length of time that they practiced the essentials of the RTM.

The actual data collection process was a modified version of an exercise called "Writing Your History (or Herstory)," traditionally used with teachers learning how to implement the RTM. The exercise has been successfully used for awareness of themselves as "mathematics" teachers (as contrasted to viewing themselves exclusively as "elementary teachers"), for getting more acquainted with one another as colleagues, and as an exercise in empathic listening (an essential in changing one's teaching, for example, from teacher-focused to student-focused). Each pair of teachers took turns in one story telling session, using a script we provided, asking each other a set of demographic questions and then inviting his or her partner to tell a story about mathematics teaching. Stories could be real or made up but had to have a hero, a villain, a turning point, and a moral. After one teacher told his or her story, the other teacher asked follow-up questions to identify and explain the "hero," "villain," "turning point," and "moral" of the story. Heroes and villains could be people, events, or things; and the moral could be the lesson learned, for example.

## Data Analysis

Transcriptions of the teachers' stories were segmented into text units and entered on NUD.IST[1] software for qualitative data management. The text units were analyzed and coded in two categories professional teacher knowledge and the Levels of Use of the RTM. The professional knowledge found in the teachers' stories addressed knowledge of the *teaching* process and themselves as mathematics teachers, knowledge of the *learning* process and of the learner, knowledge and attention to *mathematics* in the teaching practice, *pedagogy* and classroom *management*, *affect* (beliefs, attitudes, feelings related to mathematics teaching or learning), and specific reference to the *RTM*. Searches, merges, and reports are relevant evidence to our assertions and claims. A detailed report of the data analysis will be given in the research presentation.

## Discussion and Results

Within each of these text units were found evidence or implications of how teacher professional knowledge was shaped. Luminal knowledge was found between the Routine and Refinement Levels of Use. Knowledge at the routine level is defined as knowing both short and long-term requirements for use of the RTM and how to use the RTM with minimum effort. At the refinement level of use, knowledge is defined as knowing the cognitive and affective affects of the RTM on students and knowing ways for increasing impact on students.

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[1] Non-Numerical Unstructured Data Indexing Searching and Theorizing

Teachers' stories at the routine or refinement level of use of the RTM tended to emphasize the leaner and learning processes. Further analysis of text units within the luminal state of routine and refinement levels of use revealed a focus on the impact of pedagogical strategies which gave more authority to the students as opposed to a focus on teacher-centered strategies. We found teacher V-1, a second grade teacher who has been involved in the project for four years, to display a preponderance of these findings in the following text unit.

> I was able to really show enthusiasm for their [students'] thinking because I didn't have the answer. That made me more accepting of them. And by always asking another way for having gotten the answer that made my kids better thinkers because then they not only would try to figure out the answer but were able to do it differently so the they could come up and explain. Just knowing how much kids learn from each other, for somebody to come up and see it really happen when a kid explains something and another one says, oh. It was a wonderful experience for me that year and learning so much with the kids and just having see them so generally so enthusiastic about math class.

We contrast her, who is not necessarily female, with data from another teacher J-2 who is a fourth grade teacher and has been in the project for one year.

> If you are planned well enough, if you can go right through the steps, you can get excited, you don't lose your enthusiasm, and seeing that all come together gave me a renewed sense of the way things can be done.

Teacher J-2 has not reached the luminal state and is found to be at a mechanical level of use of the RTM. At the mechanical level the teacher knows on a day-to-day basis the requirements for using the RTM and is more knowledgeable of the short-term activities and effects than long-range activities and effects of the RTM.

The text units cited above from the stories of teachers V-1 and J-2 depicted patterns that were found across the data sources. We found that stories from teachers who have been engaged in RTM practice over an extended period of time were focused on the learner and the learning process, the pedagogy, and the affective impact of RTM practice on the learner and the classroom environment. Teachers' stories, given by teachers with less than two years in the project, were most often teacher-centered as portrayed in the text unit of teacher J-2.

## Conclusions

This study used storytelling as a vehicle for understanding, "What teachers know" and "How teacher knowledge is shaped by the professional knowledge context in which teachers work." Storytelling enabled the teachers to describe their changing roles and the impact of the RTM in the process. This study contributes to the growing body of research in mathematics education on the teacher's role in reform where the direction of the reform is teacher driven.

## References

Bauersfeld, H. (1992, February). The structuring of the structures. *Alternative epistemologies in education.* International Symposium, Athens, Georgia.

Clandinin, D. J., & Connelly, F. M. (1996). Teachers' professional knowledge landscapes: Teacher stories–stories of teachers–school stories–stories of schools. *Educational Researcher, 25*(3), 24-30.

Confrey, J. (1986). A critique of teacher effectiveness research in mathematics education. *Journal for Research in Mathematics Education, 17*(5), 347-359.

Confrey, J. (1992, February). The relationship between radical constructivism and social constructivism. *Alternative epistemologies in education.* International Symposium, Athens, Georgia.

Feldman. (1990). Stories as cultural creativity: On the relation between symbolism and politics in organizational change. *Human Relations*, 43(9), 809-828.

Fenstermacher, G. D. (1994). The knower and the known: The nature of knowledge in research on teaching. In L. Darling-Hammond (Ed), *Review of Research in Education, 20* (pp. 3-56). Washington, DC: AERA.

Flavell, J. H. (1979). Metacognition and cognitive monitoring: A new area of cognitive-developmental inquiry. *American Psychologist, 34*(10), 906-911.

Grouws, D., & Schultz, K. A. (1996). Mathematics teacher education. In J. Sikula, T. J. Buttery, & E. Guyton (Eds.), *Handbook for research in teacher education* (pp. 442-458). New York: Macmillan.

Hart, L. C., Schultz, K. A., Najee-ullah, D., & Nash, L. (1992). Implementing the professional standards for teaching mathematics: The role of reflection in teaching. *Arithmetic Teacher, 40*(1), 40-42.

Keys, C. W., & Golley, P. S. (1996). The power of a partner: Using collaborative reflection to support constructivist practice in middle grades science and mathematics. *Journal of Science Teacher Education*, 7(4), 229-246.

Loucks, S. F. (1983). *The Concerns Based Adoption Model.* (Series paper Number 2). Chapel Hill, NC: North Carolina University, Technical Assistance Development System. (ERIC Document Reproduction Service No. ED 233 524)

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: The Council.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: The Council.

National Research Council. (1996). *National science education standards*. Washington, DC: National Academy Press.

Project 2061, American Association for the Advancement of Science. (1993). *Benchmarks for science literacy*. New York: Oxford University Press.

Richardson, V. (1996). The role of attitudes and beliefs in learning to teach. In J. Sikula, T. J. Buttery, & E. Guyton (Eds.), *Handbook for research in teacher education* (pp. 442-458). New York: Macmillan.

Schifter, D., & Fosnot, C. T. (1993). *Reconstructing mathematics education: Stories of teachers meeting the challenge of reform*. New York: Teachers College Press.

Schoenfeld, A. H. (1987). What's all the fuss about metacognition?. In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 189-216). Hillsdale, NJ: Lawrence Erlbaum Associates.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 334- 370). New York: Macmillan.

Schultz, K. A., Hart, L.C., Najee-ullah, D., Nash, L., & Jones, J. (1993, April). *Categories of change: A study on implementing the NCTM Standards*. Paper presented at the annual meeting of the American Education Research Association, Atlanta, GA.

Thomas, C. D. (1993). *Constructivism and African-American students' confidence in mathematics*. Unpublished doctoral dissertation, Georgia State University, Atlanta.

von Glasersfeld, E. (1984). An introduction to radical constructivism. In P. Watzlawick (Ed.), *The invented reality* (pp. 17-40). New York: Norton.

von Glasersfeld, E. (1990). An exposition on constructivism: Why some like it radical. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics (Journal for Research in Mathematics Education Monograph No. 4*; pp. 19-30). Reston, VA: National Council of Teachers of Mathematics.

Wagner, D. A. (1994). A comparative study of Atlanta Math Project and future Atlanta Math Project teachers' questioning and interactive decision-making strategies. Unpublished doctoral dissertation. Georgia State University, Atlanta.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27*(4), 458-477.

# TEACHERS' BELIEFS ABOUT MATHEMATICS AS ASSESSED WITH REPERTORY GRID METHODOLOGY

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Seventeen teachers enrolled in a summer class for teachers in bilingual settings were asked to provide descriptions of "typical things a person does when he or she does mathematics." From their responses we selected items for use in repertory grids. A sample of three teachers were then given these items and constructs were elicited. Representative constructs were chosen from those obtained, and each of the 17 teachers completed a repertory grid designed to illuminate their conceptions of what it means to do mathematics. Grids were analyzed using fuzzy implications, and the resulting implication structures were then classified according to their structural sophistication. The classifications related strongly to self-reported experience with mathematics, and give insight into the relationship between teaching and views of mathematics.

## Background

It has been pointed out repeatedly that teaching mathematics is based on beliefs about what mathematics is (Thompson, 1992; Hersh, 1986). Recent works in the philosophy of mathematics point out changing conceptions of mathematics and how these conceptions relate to reform efforts in mathematics education (Ernst, 1991; Tymoczko, 1986; Restivo, Van Bendegem, & Fischer, 1993). Although Thompson's (1992) review of the literature finds varying degrees of consistency between professed beliefs and classroom instruction, it seems clear that beliefs about mathematics will affect decisions teachers make as they attempt to alter their classroom practice. As Hersh (1986) points out, "One's conceptions of what mathematics *is* affects one's conceptions of how it should be presented. One's manner of presenting it is an indication of what one believes to be most essential in it" (p. 13). This seems particularly important as teachers make changes in their practice to accommodate students whose primary language is not the majority language of the class. How teachers adapt instruction will depend in part on what they see as being fundamental to the mathematics they teach. It thus becomes critical to help identify teachers' beliefs about mathematics.

Thompson's (1992) review makes it clear that most work on teachers' beliefs about mathematics has been based on individual interviews or questionnaires. The repertory grid methodology employed in this study has been used

113

in the past to assess teachers' beliefs in various domains, but only recently have knowledge of mathematics or beliefs about mathematics been addressed in this way (Lehrer & Franke, 1992; Middleton, 1995).

Repertory grid methodology was introduced as a psychological assessment tool by George Kelly in the mid-1950's. Originally, subjects were asked to produce a list of people who filled various roles in their lives (e.g., a trusted friend). Having produced the list, they were given items from this list three at a time and asked to describe how two were alike and how they differed from the third. The categories arising from this task were called *constructs* and were thought to be the organizing templates used by subjects to give meaning to their interpersonal world. These constructs were bipolar so that every construct (e.g. *honest*) had an opposing *contrast* (e.g. *dishonest*). Thus there is always a continuum defined between a construct and its opposing contrast. In this study, the constructs used by subjects to organize items dealing with mathematics are considered to be the organizational templates used to make sense of mathematical activity, and to reflect subjects' beliefs about what mathematical activity is.

## Data Collection

Subjects for this study were 17 public school teachers enrolled at a major northwestern university in a summer course designed to help them teach mathematics in a bilingual setting. Some of the teachers were themselves bilingual; many were not. Each member of the class was asked to provide a list of "typical things a person does when he or she does mathematics." From these lists, eighteen of the most representative answers were chosen as items for eliciting repertory grids. This list of 18 items included, "ask or answer questions; compute or calculate; discover, explore or experiment; find patterns; get frustrated; solve problems; think or reason; talk to others; visualize; work alone." With these 18 items, constructs were elicited from three volunteers from the class. These volunteers met with a member of the research team and were given items from the list three at a time, and asked to describe how two were the same and therefore different from the third. In this way, constructs and contrasts were elicited. The thirteen most common and representative constructs were chosen from among those elicited, and each class member was then asked to rate every item on a three-point scale as being more like the construct or its contrast. The thirteen construct/contrast pairs were:

- Process/Outcome
- Concrete/Abstract
- Looks at Whole/Looks at Parts
- Creative/Constrained
- Higher Level/Rote

- Easy/Difficult
- Students' Acts/Teachers' Acts
- Inviting/Threatening
- Discovering/Following Rules
- Interacting/Working Alone
- Rewarded/Unrewarded
- Physically Active/Physically Passive
- Emotionally Involved/Emotionally Detached

The result for each class member was a matrix of ratings of each item on each construct/contrast pair. If each construct is seen as describing a class of items, the ratings can be seen as giving the degree to which each item is a member of that class. More formally, if these classes are viewed as fuzzy sets, then the rows of the matrix provide the degree of membership of each item in the fuzzy set named by the construct. On this basis Gaines and Shaw (1986) demonstrated how implications can be defined between the predicates which define these fuzzy sets (in this case, the constructs). For example, if the rankings for all items on the construct "abstract" are as strong or stronger than for the construct "rote," it indicates that whenever an item is seen as rote it is also seen as abstract. Thus, an item's being rote implies its being abstract, and we conclude that rote implies abstract (rote ø abstract) in general. Gaines and Shaw describe a process for establishing such implications which guards against the inclusion of "trivial" implications and which also finds bi-implications where, for example, rote ø abstract, and abstract ø rote.

### Results

On the basis of the complexity of the implication structures, class members were divided into two major groups. Each is described briefly here. The first group consisted of five teachers with scant and simple implication structures. Of these, one teacher had no implications, and the others had no more than two implications. Examples of typical implications are "rote ø constrained" and "rote ø following rules." Both of these implications make sense and demonstrate an observed tendency for implications to deal more with aspects of mathematics which might be perceived as negatively oriented toward reform. All five teachers in this group identified themselves as having a relatively weak math background.

The second group consisted of teachers who displayed more than two simple implications or who displayed more complicated chains of implications, such as "physically active ø discovering ø higher level." Seven of these teachers had either several implications or implication chains of this type. Also within this group were five teachers who had much more complex patterns. Of these five, four reported having relatively strong mathematics backgrounds. Typical of this subgroup were teachers who produced interrelated implications involving the majority of the 13 constructs, as well as bi-implications. These more complicated structures give some insights into individual teachers' beliefs about mathematics. We provide two brief examples here.

The implication structure produced by one of our teachers was as follows:

- Process ø Students' Acts ø Following Rules ø Working Alone
- Process ø Students' Acts ø Threatening ø Difficult
- Process ø Students' Acts ø Emotionally Detached ø Threatening ø Difficult
- Process ø Students' Acts ø Working alone
- Higher Level ø Abstract ø Threatening ø Difficult
- Higher Level ø Looks at whole ø Threatening ø Difficult
- Higher Level ø Threatening ø Difficult

Here, all but two of the implication chains end with the implication threatening ø difficult. For this teacher, it seems that a great deal of mathematics is both threatening and difficult, including that which involves students acting (as opposed to teachers being the major actors), that which is higher level (as opposed to rote), and that which involves a focus on process rather than on product. This teacher reported having a relatively strong background in mathematics, but the beliefs about mathematics which are exhibited here have some troublesome implications for instruction. For example, to avoid making mathematics threatening or difficult for students, this teacher may well choose to focus largely on tasks which focus on products rather than processes, and which are rote, concrete, and narrow.

By contrast, a second teacher, who also reported having a strong background in mathematics, had the following implication structure:

- Outcome ø Difficult ø Following Rules ø Threatening ø Abstract
- Outcome ø Difficult ø Following Rules ø Abstract
- Outcome ø Difficult ø Physically Passive ø Abstract
- Outcome ø Difficult ø Threatening ø Abstract
- Outcome ø Difficult ø Working Alone ø Abstract
- Outcome ø Difficult ø Abstract
- Outcome ø Following Rules ø Threatening ø Abstract
- Outcome ø Following Rules ø Abstract
- Outcome ø Emotionally Detached ø Threatening ø Abstract

- Outcome ø Emotionally Detached ø Abstract
- Unrewarded ø Discovering ø Emotionally Involved
- Physically Active ø Discovering ø Emotionally Involved
- Physically Active ø Emotionally Involved
- Physically Active ø Process

In addition, bi-implications indicated that for this teacher, seven constructs were very closely related: Working Alone, Following Rules, Threatening, Physically Passive, Rewarded, Emotionally Detached, and Difficult. Together with the implications above, this seems to indicate that instruction based on outcomes, as opposed to processes, is seen as rule-based, solitary, emotionally detached, passive, difficult, and (regrettably) rewarded in traditional schooling. In addition, all of this is associated with "abstract" mathematics. On the other hand, another cluster of implications, and their contrapositives (which involve the contrasts rather than the constructs) seem to indicate that for this teacher, physically active tasks are seen as inviting, focused on process, involving in discovery, and less difficult. This cluster indicates the teacher's acceptance of an alternative to the traditional instruction which is viewed in such a negative way.

## Conclusions

One of the major goals of the course in which these teachers were enrolled was to suggest that good mathematics instruction for all students can be richly contextual, activity-based, and engaging, and that this approach supports students whose primary language is not the language of instruction. In light of this goal, and in light of the general efforts toward reform, the beliefs exhibited by some of the teachers were somewhat troubling. Although we do not suggest that either of the two teachers we discuss here have wholly positive or negative perceptions of mathematics, it is clear that they point to some areas of concern.

The repertory grid methodology employed exhibited some specific strengths and weaknesses. For those teachers whose implication structures were relatively complex, the method provided easily interpretable, meaningful information about beliefs. Moreover, it allowed for aspects of mathematical practice to emerge from the class itself, rather than being imposed by us as researchers. However, for some teachers it gave very little information. It is probable that refinements in the methodology could somewhat improve results, but it also seems likely that this methodology should be supplemented by other data sources.

## References

Ernest, P. (1991). *The philosophy of mathematics education.* London: Falmer Press.

Gaines, B. R., & Shaw, M. L. G. (1986). Induction of inference rules for expert systems. *Fuzzy Sets and Systems, 18,* 315-328.

Hersh, R. (1986). Some proposals for reviving the philosophy of mathematics. In T. Tymoczko (Ed.), *New directions in the philosophy of mathematics: An anthology* (pp. 9 - 28). Boston: Birkhauser.

Lehrer, R., & Franke, M. L. (1992). Applying personal construct psychology to the study of teachers' knowledge of fractions. *Journal for Research in Mathematics Education, 23*(3), 223-241.

Middleton, J. A. (1995) A study of intrinsic motivation in the mathematics classroom: A personal constructs approach. *Journal for Research in Mathematics Education, 26*(3), 254-79.

Restivo, S., Van Bendegem, J. P., & Fischer, R. (1993). *Math worlds: Philosophical and social studies of mathematics and mathematics education.* Albany, NY: SUNY Press.

Thompson, A. (1992). Teachers' beliefs and conceptions: a synthesis of research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127-146). New York: Macmillan.

Tymoczko, T. (1986). *New directions in the philosophy of mathematics: An anthology.* Boston: Birkhauser.

# THE GEOMETRY CLASSROOM: THE INFLUENCE OF TEACHERS' BELIEFS

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This paper presents partial results of a study that investigated secondary geometry teachers' decision making in a mathematics curricular reform context. The study examined teachers' planning and interactive decisions and identified factors that influenced the decisions that these teachers made. Results reported include the description of the geometry courses generated by teachers' planning and interactive decisions and the discussion of one of the identified factors—teachers' beliefs.

Teachers' planning and interactive decisions as well as the identification of factors influencing teachers' decision making have been recognized an important components for understanding teaching (Borko & Livingston, 1989; Brown & Borko, 1992; Clark & Peterson, 1986; Hsieh & Spodek, 1995; Putnam, 1984; Westerman, 1991). To better understand geometry teaching, this exploratory investigation focused on secondary geometry teachers' planning and interactive decisions and the context in which these decisions were made. The contextual factor that is the focus of this paper is teachers' beliefs about the nature of geometry as a discipline and about their view of their own teaching.

## Design and Method

The sample consisted of five mathematics teachers, three females and two males, from five high schools (grades 9 through 12) located in the Northwestern United States. The subjects' geometry teaching experience ranged from 8 years to 23 years ($M = 15.2$). Prior to the start of the school year, data were collected through a pre-questionnaire and pre-interview. Beginning with the start of the school year and continuing for three months, weekly audiotaped and videotaped classroom observations and audiotaped teacher interviews were conducted. In addition, written documents such as weekly lesson plans, lesson plans for observed lessons, textbooks, and assessment instruments were collected. At the conclusion of the observation process, a final interview was conducted with each teacher.

The data were collected and analyzed in three phases using the ongoing, inductive process described by Bogdan and Biklen (1992). This general inductive process involved reading the data, organizing it, breaking it into manageable units such as codes or categories, searching the data for patterns, refining categories as needed, and identifying themes that were prevalent in the decision making data. The first phase focused on the analysis of the pre-academic-

124

year questionnaire responses and interview transcripts. The results of this analysis guided the data collection and analysis occurring in the second phase. The second phase of analysis began with the first observation of each teacher and continued through each teacher's final interview. After each teacher's observed lesson, the videotape of the lesson was viewed and notes were added to the field notes for that lesson. Following each observed lesson, the interview notes, the field notes that contained information from the videotapes, and written documents were analyzed.

During the third phase of analysis, the audiotape transcripts were merged with the field notes of each lesson in order to make the transcript for each lesson complete. A preliminary profile consisting of the teacher's decisions and influential factors was prepared. To complete the analysis, each teacher's data were re-examined. The focus of the re-examination was on identifying situations that did not support existing categories. When this circumstance occurred, the category was refined. Based on the analyses, individual teacher profiles were written. Teacher profiles were then examined collectively in search of similarities and differences across the sample, generating a cross-case profile.

## Results and Discussion

Teachers' beliefs about the nature of geometry were related to the facets of geometry promoted in their classrooms. One facet that was valued by all teachers was geometry as a content knowledge base. This knowledge base included geometry concepts (e.g., points, lines, planes, angles, polygons, three-dimensional figures) and relationships between these concepts. Teachers also promoted geometry as an example of a mathematical system. This facet included both the logical development of geometry and the process of how mathematicians derive mathematics. One teacher promoted the logical development of a proof-oriented geometry course through the use of lectures. In contrast, the other four teachers' interpretation of the mathematical system view indicated that the development of deductive reasoning skills would be one expected outcome of geometry, not the primary focus of their course. For three of the four teachers, proofs via deductive reasoning were one component of the process of developing mathematical concepts and relationships through exploring, conjecturing, and verifying geometric ideas. The system aspect of geometry promoted by these three teachers was in agreement with the *Curriculum and Evaluation Standards for School Mathematics* (1989, hereafter referred to as the *Curriculum Standards*) idea that the deductive perspective of geometry needed to receive less emphasis than in the past and the interplay of inductive and deductive reasoning needed to be fostered more in present and future geometry classes. Based on her encouragement of student development of inductive and informal deductive reasoning skills, the remaining teacher appeared to be moving toward this same interpretation of the system aspect of geometry.

Another facet of geometry addressed by the teachers included geometry as a setting for developing communication and problem solving skills. For the proof-oriented teacher, students had some opportunity to communicate geometric ideas as they volunteered to write their proofs on the board and explain their work to their classmates. For the other four teachers, geometry as a setting for developing communication skills was evident as their students were encouraged to read, write, and talk about geometry concepts and relationships. In agreement with the *Curriculum Standards*' (1989) description of communication, the focus of these communication processes was on student understanding of geometric ideas. The geometry courses of these four teachers were also seen as a setting for developing problem solving skills such as looking for patterns, drawing and interpreting diagrams, performing and applying constructions, and exploring concepts and relationships through investigations. These mathematical skills which advocated students' active involvement in "doing mathematics" supported teachers' efforts toward integrating a problem solving approach for learning mathematics as described by the *Curriculum Standards*. The fourth theme of the *Curriculum Standards*, promoting mathematical connections, was hinted at as teachers used real-world representations for vocabulary or when they used cartoons to explore a topic.

Teachers' beliefs about the nature of geometry were characterized by the different emphases given to the facets of geometry. One teacher emphasized geometry as an example of a mathematical system as well as addressed geometry as a base of specific content knowledge via a lecture approach. This teacher appeared to believe that geometry was a static and structured body of knowledge and procedures that students must master. The other four teachers emphasized geometry as a knowledge base, an example of a mathematical system, and as a setting for developing communication and problem solving skills while students actively participated in their learning of geometry. These four teachers also incidentally addressed the connection between geometry and the real world. For these four teachers, their views suggested that they believed that geometry was a multifaceted body of knowledge that needed to be examined, explored, and constructed by students.

The depiction of geometry by four of the five teachers was similar to geometry researchers' previous descriptions of geometry. Burger and Culpepper (1993) described geometry as an abstraction of visual and spatial experiences, as a provision of approaches for problem solving, and as an environment for studying mathematical structure. Usiskin (1987) characterized geometry as the visualization, construction, and measurement of figures, as the study of the real, physical world, as a vehicle for representing other mathematical concepts, and as an example of a mathematical system.

Teachers' beliefs about their view of their own teaching also influenced teachers' decision making. One aspect of this belief was whether teachers saw the process of becoming an effective mathematics teacher as a life-long process. In self-descriptions of their past and present teaching, three teachers implied that their teaching had changed since they first started teaching. In addition, their comments indicated they continued to explore ways to provide their students with better opportunities for learning geometry. These three teachers appeared to view the process of becoming an effective geometry teacher as a life-long process. A fourth teacher's description of his teaching suggested that his teaching had also changed since his first teaching position. His comments also indicated that he was open to seeking out new ways to teach geometry, but at the same time he was satisfied with his course's format. His openness to future change indicated he probably viewed the process of becoming an effective geometry teacher as a life-long process. The fifth teacher's description of his teaching implied that he had found an instructional approach early in his career with which he was comfortable, and he continued to use that approach. Thus, it appeared that this teacher might not view the process of becoming an effective geometry teacher a life-long process. This study's finding of whether teachers view themselves as life-long learners supported Hsieh and Spodek's (1995) finding that teachers' decision making was influenced by teachers' vision of themselves as teacher.

Teachers' beliefs about their view of their own teaching also involved whether teachers appeared to think of themselves as curriculum builders. As curriculum builders, teachers defined their own geometry courses rather than implemented courses defined by others. Teachers in studies conducted by Putnam (1984) and Westerman (1991) took an active role in making decisions that transformed their curriculum into their desired course. For the teachers in this study, the role assumed by them in the process of defining their geometry course was influenced by whether they chose their textbook and whether their textbook matched their beliefs about the teaching and learning of geometry. Two of the teachers relied heavily on the textbook for content selection and sequencing. For both of these teachers, the reliance on their textbook was attributed to the fact that they helped select a textbook that matched their philosophies for teaching geometry. Even though a third teacher also used a textbook that she helpeḋ select and that supported her approach for teaching geometry, the textbook did not completely define the her desired geometry course. This teacher supplemented her course with activities and projects that supported an integrated mathematics class and her school's restructuring plan. The remaining two teachers used the textbook as a guide and made decisions about their course based on their wish to present a view of geometry that was broader than the proof-oriented scope and sequence emphasized in their textbooks. Both teachers participated in the book selection process at their respective schools.

Their agreement to select a textbook that did not match their philosophies for teaching geometry appeared to be related to their beliefs that their geometry course was not defined by the textbook.

## References

Bogdan, R. C., & Biklen, S. K. (1992). *Qualitative research for education* (2nd ed.). Boston, MA: Allyn and Bacon.

Borko, H., & Livingston, C. (1989). Cognition and improvisation: Differences in mathematics instruction by expert and novice teachers. *American Educational Research Journal, 26*(4), 473-498.

Brown, C. A., & Borko, H. (1992). Becoming a mathematics teacher. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 209-239). New York: Macmillan.

Burger, W. F., & Culpepper, B. (1993). Restructuring geometry. In S. Wagner (Ed.), *Research ideas for the classroom: Vol. 3. High school mathematics* (pp. 140-154). New York: Macmillan.

Clark, C. M., & Peterson, P. L. (1986). Teachers' thought processes. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (3rd ed., pp. 255-296). New York: Macmillan.

Hsieh, Y., & Spodek, B. (1995). *Educational principles underlying the classroom decision-making of two kindergarten teachers.* Taiwan. (ERIC Reproduction Service No. ED 383 663)

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: The Council.

Putnam, J. G. (1984). *One exceptional teacher's systematic decision-making model.* East Lansing, MI: Michigan State University. (ERIC Reproduction Service No. ED 242 702)

Usiskin, Z. (1987). Resolving the continuing dilemma in school geometry. In M. M. Lindquist & A. P. Shulte (Eds.), *Learning and teaching geometry, K-12, 1987 yearbook* (pp. 17-31). Reston, VA: National Council of Teachers of Mathematics.

Westerman, D. A. (1991). Expert and novice teacher decision making. *Journal of Teacher Education, 42*(4), 292-305.

# CHANGE IN TEACHING PRACTICE DURING AN ACTION RESEARCH COLLABORATIVE

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Miller and Pine (1990) define action research as a procedure by which teachers examine the process of teaching and learning in their own classrooms. Clift and others (1988) characterize collaborative action research as focusing on practical problems of individual teachers as they interact with university staff. While clearly providing a context for collaboration and support as teachers attempt to change teaching practices, such collaboratives are also rich in their potential for inducing a reflective teaching practice (Raymond, 1996).

The authors, a middle school mathematics teacher and a teacher educator, have engaged in a two-year action research collaborative. Throughout this collaboration, they have acted on an equal footing as colleagues, rather than as university professor and middle school teacher. The collaboration has included a narrative, written by the teacher, describing her work and the changes she has made, or attempted to make, in her practice of teaching mathematics. While the formal writing of a narrative description of one's own work provides a context for reflection on that work, this activity may also provide a context for ongoing support of the teacher's efforts to change (Schifter, 1996).

The narrative provides a small window through which we might view, from her perspective, the process of change in one middle school teacher's mathematics teaching practice. Doing so may help us understand more clearly her process of change, and how that process might unfold in others.

## References

Clift, R., Veal, M., Johnson, M., & Holland, P. (1990). Restructuring teacher education through collaborative action research. *Journal of Teacher Education, 41*(2), 52-62.

Miller, D. M., & Pine, G. J. (1990). Advancing professional inquiry for educational improvement through action research. *Journal of Staff Development, 11*(Summer 1990), 56-61.

Raymond, A. (1996). What is collaborative action research? In A, M. Raymond, C. D. Rafferty, & K. M. Dutt (Eds.), *Collaborative action research: Case studies of school-university initiatives* (pp. 1-7). Terre Haute, IN: Curriculum Research and Development Center, School of Education, Indiana State University.

Schifter, D. (1996). A constructivist perspective on teaching and learning mathematics. In C. T. Fosnot (Ed.), *Constructivism: Theory, perspectives, and practice* (pp. 73-91). New York: Teachers College Press.

# PRE-SERVICE K-12 TEACHERS OF MATHEMATICS: CHANGE IN BELIEFS

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Piaget characterizes learning as a continual process of assimilation and accommodation (Wadsworth, 1984). Attitudes and beliefs about mathematics and mathematics content knowledge of teachers are important in the preparation of teachers of mathematics (Thompson 1992).

This paper describes some of the changes that pre-service K-8 or 7-12 teachers experience during the teacher preparation program. The research focused on students' understanding and beliefs about "What is mathematics?" and "What does it take to be a successful teacher of mathematics?" Initial responses were e-mailed to the methods instructor on the first day of class before any discussion of the course philosophy or objectives. A final interview of purposefully selected students occurred at the end of the course. Analysis of data from the initial e-mail survey, reflective journals submitted throughout the course, and the final interview indicate students' beliefs about teaching and learning mathematics changed as a result of a combination of course work and maturation.

Results indicated content courses "take off the blinders to mathematics" (Tim, #2); the hands-on approach helped students make mathematical connections. Students indicated they had a reason to learn the mathematics: They wanted to be able to teach it. "They need a strong understanding of the subject matter, . . . the teacher needs to be willing to learn and change as time goes by, . . . [and] be very flexible" (SS#3.1).

As pre-service teachers proceed through the course of study to become certified teachers, they continually refine and create new understandings of what it means to be a teacher of mathematics and the nature of mathematics itself. This study documents those beliefs and changes in beliefs of students at one university.

## References

Thompson, A. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 127-146). New York: Macmillan.

Wadsworth, B. J. (1984). *Piaget's theory of cognitive and affective development*. New York: Longman.

# MATHEMATICS PROFICIENCY AS VIEWED THROUGH A NOVICE TEACHER'S GOALS, BELIEFS, AND ORIENTATIONS

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A two year case study of novice teacher Monica's goals, beliefs, and orientations revealed that a positive sense of mathematics proficiency and a strong display of content knowledge can negatively influence the professional development of a mathematics teacher. Within constructivist and interactionist frameworks, Monica and I collaboratively researched how she was coming to know herself as a mathematics teacher. She entered a secondary mathematics teacher education program with strong credentials and a commitment to mathematics teaching. Midway in her first year of teaching, she considered leaving the profession. By year's end she had found her place and had dismissed as unrealistic, the methodologies promoted in her university program.

The interview, observation, and focus survey data and the constant comparative and retrospective analyses provided insights into Monica's sense of mathematics proficiency. She evidenced a combination of authoritative, utilitarian, and mathematics-centered belief systems (Ernest, 1991). Mathematics was a structured collection of definitions and algorithms that were useful in real life situations. Mathematics was absolute, certain, and God-given. The teacher's responsibility was to present clear definitions and step by step examples. *Teaching* a group of disinterested students was problematic for Monica, but *mathematics* teaching was unproblematic.

Monica's experiences with investigative learning environments during her teacher education program failed to connect with her view of mathematics, her needs as a received knower, and her performance goal orientation. She believed that an inquiring disposition distracted one from doing mathematics. Mathematics proficiency was evidenced by answering questions asked of a teacher. The research findings support the recommendation that mathematics teacher educators strive to help novice teachers build from the novice's personal theories and in so doing ascertain not only *what* mathematics they know but *how* they come to know it.

## Reference

Ernest, P. (1991). *The philosophy of mathematics education.* Bristol, PA: Falmer Press.

# DEVELOPING SELF CONFIDENCE IN MATHEMATICS: AN EXPERIMENT WITH PRESERVICE ELEMENTARY SCHOOL TEACHERS

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One of the current goals of mathematics education is to have students develop confidence in their ability to do mathematics (NCTM, 1989). However, many elementary school teachers fear mathematics and lack self confidence in the subject. This situation raises questions about whether teachers, with little or no confidence in their own abilities, can facilitate students' development of self confidence in mathematics. This paper reports on a teaching experiment designed to help preservice teachers develop self confidence as they acquire content and pedagogical knowledge of mathematics.

## Theoretical Background

Hilton (1980a,b) suggests that mathematics anxiety is caused by the lack of adequate knowledge of mathematics. He recommends giving students proper grounding in mathematics and engaging mathophobic adults in mathematical reasoning tasks. Bloom (1972) argues that most students can be helped to achieve at a high level by using mastery learning strategies in teaching. These ideas guided the teaching experiment described in the paper.

## Procedures and Results

The study involved 22 elementary education majors enrolled in a mathematics "methods" course. While learning mathematics pedagogy, the students worked on filling gaps in their content knowledge. They explored non-routine problems every week and gave written solutions and reflections on their work. Twenty one students reached a mastery level of at least 80% on a comprehensive elementary mathematics test. They showed greater self confidence by their willingness to take on tough math problems and persist in their search for solutions.

## References

Bloom, B. S. (1971). *Individual differences in school achievement: A vanishing point?* Bloomington, IN: Phi Delta Kappa, Inc.

Hilton, P. (1980 a). Math anxiety: Some suggested causes and cures, Part 1. *Two-Year College Mathematics Journal, 11*(13), 174-188.

Hilton, P. (1980 b). Math anxiety: Some suggested causes and cures, Part 2. *Two-Year College Mathematics Journal, 11*(4), 246-251.

National Council of Teachers of Mathematics (NCTM). (1989). *Curriculum and Evaluation Standards for School Mathematics.* Reston VA: NCTM.

# THE INFLUENCE OF BELIEFS ON BEHAVIOR AND THUS ON LEARNING

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Affective factors, referring to a wide range of feelings and moods differing from pure cognition (McLeod, 1989) are an indispensable part of mathematical learning. The affective emphasis of this discussion will be students' beliefs about themselves as learners. Silver (1987) contends that students could potentially gain more from instructors' attention to the curriculum of beliefs than from any improvement of mathematical facts.

The basis for this discussion will be the theory of axiom change (McEntire & Kitchens, 1984) which states that *experiences* lead to the formation of *beliefs* (axioms) and thus affect *behavior*. Several examples from the mathematics classroom will illustrate that beliefs can dictate behavior and can mask mathematical ability . Therefore, teachers must shift focus from informally determining students' ability based on their performance to exploring students' beliefs.

To explore the effects of self-beliefs on performance, we propose to discuss the following issues:

1. The relationship among students' experiences, beliefs, and behaviors.
2. The facilitation of students' exploration of self-beliefs by math instructors, untrained in psychology.
3. The impact of instructor beliefs, especially the one "all students in the class can learn".
4. The relationship between the instructor's high cognitive expectations and students' development of positive beliefs.

Participants will be encouraged to relate the theory of axiom change and its implications for the teaching of mathematics to examples both from their own classroom and from their own lives. Each individual teacher should leave this discussion with new ideas for becoming actively involved in helping students to understand the effect of self-beliefs on learning and with a model from which to operate in helping students.

## References

McEntire, A., & Kitchens, A. N. (1984). A new focus for educational improvement through cognitive and other structuring of subconscious personal axioms. *Education, 105*(2), 139-146.

McLeod, D. B. (1989). Beliefs, attitudes, and emotions: New views of affect in mathematics education. In D. B. McLeod & V. M. Adams (Eds.), *Affect in mathematical problem solving: A new perspective* (pp. 245-257). New York: Springer-Verlag.

Silver, E. A. (1987). Foundations of cognitive theory and research for mathematics problem-solving instruction. In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 36-60). Hillsdale, NJ: Lawrence Erlbaum.

# TEACHER EDUCATION

# MATHEMATICS STUDENT TEACHERS' DEVELOPMENT OF TEACHER KNOWLEDGE AND REFLECTION

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Prospective mathematics teachers' perceptions of student teaching in their development as teachers including their development of teacher knowledge and reflection were investigated. Data collection included written documents from student teaching and in-depth interviews. Four areas of perceived knowledge development surfaced through the data analysis: general teaching knowledge, mathematics teaching knowledge, knowledge about students, and knowledge of the context. Van Manen's (1977) levels of reflectivity (i.e., technical, practical, and critical) were evidenced in the teachers' oral and written reflections. Their level of reflectivity was influenced by the ability to experiment and the support received for reflection. Connections between reflection, teaching, and knowledge development were sought.

Teacher knowledge is unquestionably one of the most important influences on what mathematics teachers do in classrooms and consequently on what students learn. In recent years, researchers have called attention to the need for research on preservice teachers' development of knowledge during teacher education (Grossman, 1990; Lanier & Little, 1986). Concurrently, researchers have sought the identification and characterization of domains of knowledge that comprise a professional knowledge base for teaching (Fennema & Franke, 1992; Reynolds, 1989; Shulman, 1987). Reports of the Holmes Group (1986) and the Carnegie Task Force on Teaching as a Profession (1986) advocate the importance of a knowledge base in framing teacher education and practice. In addition to developing domains of teacher knowledge, learning to teach necessitates the development of cognitive processes for teaching (Brown & Borko, 1992). In particular, the development of reflective teachers has become a prominent issue in discussions of teacher education.

Empirical research is needed to support and guide teacher education programs seeking to promote the development of teacher knowledge and reflection among prospective mathematics teachers. The purpose of this study was to research prospective mathematics teachers' perceptions of their development of teacher knowledge and reflection during their mathematics teacher education program. Experiences prior to and during student teaching that prospective teachers identified as sources influencing their development were investigated in order to better understand the transition from mathematics student to mathematics teacher. In particular, this study revealed important connections among reflection, teaching, and supervision in prospective teachers' development of teacher knowledge during student teaching.

## Theoretical Perspective

The study reported here investigated the development of teacher knowledge and reflection given the following perspective. Knowledge construction was assumed to be a dynamic and active process (von Glasersfeld, 1984). Although learners make sense individually, this process is influenced by the experiences in which they are engaged including interactions with others. The construction of knowledge involves the interaction of past knowledge with the experience of the moment. Reflection is a deliberate act in the process of making sense of one's experiences. Given the assumption that learners construct knowledge while making sense of their experiences, reflection is important in the process of learning (Baird, 1992). Products of this process in the form of written or oral reflections reveal differences in individuals' reflections. Van Manen's (1977) conception of "levels of reflectivity" addresses differences in reflection. Van Manen outlined three levels of reflection: technical, practical, and critical. The technical level is concerned with the efficient application of educational knowledge to attain given ends. The practical level is concerned with assessing the educational consequences toward which an action leads. The critical level is concerned with the application of ethical and moral criteria to the assessment of those consequences.

## Design of the Study

A case study methodology was used in this investigation in an attempt to gather in-depth data about a specific phenomena (Merriam, 1988): mathematics student teachers' knowledge and reflection. The design of this study consisted of two case studies of prospective teachers enrolled in the same upper division mathematics teacher education program. Data collection for each case included documents developed during the 10 weeks of student teaching and three 2 hour interviews conducted at the end of student teaching, during a 2 month period. The data analysis began with the documents: student teaching journals, student teaching summative reflection papers, classroom observation field notes, and notes from school visits and other interactions with the student teachers. Part of the first interview addressed questions and issues that arose from the documents. Subsequent interviews were guided by the ongoing analysis. Each case provided a unit of analysis through which themes and patterns were identified that were used in the cross-case analysis.

## Prospective Teachers

Ginger and Maryanne each excelled in their mathematics and education coursework and were regarded highly by the mathematics education faculty. Prior to student teaching, they were enrolled in the same two-quarter sequence of core courses concerned with curriculum and instruction for teaching secondary school mathematics. During their student teaching, I was their university supervisor. Both taught students in traditionally college bound and non-college bound mathematics courses.

Student teaching gave Ginger "a taste of teaching;" however, she felt it had been "an endurance test." She struggled to develop as a teacher within the set routines of her cooperating teacher, Ms. Spice. Ms. Spice typically relied on teacher-centered instruction devoid of varied materials or tools for discourse. Ms. Spice was not supportive of Ginger's experimentation and did not help Ginger reflect on her teaching.

Ginger's reflections during student teaching were mainly at the technical level. She often focused on classroom management strategies without acknowledging the educational consequences for students' learning of the mathematics or considering the social conditions and worthwhileness of that knowledge. For example, she let students work together on class work "because when they worked alone they got bored" and she prepared class notes on transparencies because "it made me more mobile, it helped with classroom management some, and um I could see if they were having problems." Ginger's reflection at the practical and critical levels occurred when she felt comfortable experimenting and felt supervisory support. She experimented on days I observed and reached higher levels of reflection through our discussions. For example, on one of those occasions, she wrote that lesson "would have been more conducive to learning & mathematical discussion if I'd grouped the conics that were the same together, then students could have compared."

Given the lack of feedback, Ginger began to rely heavily on students' comments without critically reflecting on reasons for those comments. Some students told her they preferred lecture over group work and a few said they did not like to use calculators. These preferences cinched for Ginger that she needed to use a variety of teaching strategies to accommodate student preferences. In a final evaluation, her students indicated that she needed to work on classroom management. Ginger felt her most important learning from student teaching was that "students need to be in a classroom with effective classroom management." That means she needs to keep students busy because "there are less problems with classroom management [when] the students are not allowed idle time or idle minds."

Student teaching gave Maryanne an opportunity "to try out everything or . . . some of the things you've been learning about." Maryanne's cooperating teacher, Mrs. Gilligan, used a variety of strategies and materials for teaching mathematics. Her teaching was aligned to some extent with that promoted in the teacher education program. Mrs. Gilligan was very supportive of Maryanne, giving her freedom and support to explore varied teaching strategies.

Maryanne's reflections during student teaching were primarily at the practical level. She often focused on the educational consequences of her actions (i.e, students' learning of mathematics and learning to work in groups). For example, she commented in her journal, "Their group skills are a little weak. I'm trying to make suggestions and adjustments as I put them in groups." Mrs. Gilligan and Maryanne engaged regularly in reflective discussions of their teaching. According to Maryanne, Mrs. Gilligan helped her learn "how to self-evaluate." During my post-observation discussions with Maryanne, I was able to help her reflect at the critical level, considering ethical outcomes of her actions. After a discussion of a videotape of her teaching, she wrote, "I would like there to be even more student talking (to each othe). . . . I will be working on them not just looking to me for the answers." Maryanne's technical level reflections were typically associated with her teaching of trigonometry. In the trigonometry classes, Maryanne felt less freedom to experiment " maybe she [Mrs. Gilligan] wanted to keep an eye on what was going on in her trig." However, Maryanne also felt less familiar with the mathematics in that course.

Maryanne felt her most important learning from student teaching was that "students can work through things that are difficult for them and eventually 'get it'." Also important to her was learning "what things I need to consider when I am thinking about a [mathematics] lesson." For Maryanne, continued learning of how to teach mathematics was very important: "I see myself as trying a lot, you know, not just saying, 'Oh, okay, that didn't work, I think I'll lecture . . . but saying, "It didn't work. Why didn't it work? Let's try again or let's try something else and see if this works better."

## Findings

The cases of Maryanne and Ginger provide much insight into the development of mathematics teachers including factors influencing that development. They suggest that in order for student teaching to enhance prospective teachers' development as cognizing and reflective practitioners, the experience needs to be perceived by student teachers as an opportunity to experiment with knowledge they have been developing in their teacher education program and as an opportunity to reflect on that experimentation with support from supervisors.

Maryanne and Ginger felt that through student teaching they expanded the knowledge they had developed in their prior teacher education coursework. Both perceived their development of knowledge in four areas: general teaching knowledge, mathematics teaching knowledge, knowledge about students, and knowledge of the context. However, the extent of knowledge developed in each of these areas was mitigated by their ability to experiment and reflect on their teaching and the support they received in these areas from their supervisors. For Ginger, her experience precipitated the centrality of classroom management and accommodating students' preferences as the most important factors to consider in her teaching of mathematics. Maryanne's experience focused her attention on thinking about the mathematics teaching and students' learning as central factors to consider in her teaching.

This study revealed that prospective teachers generate oral and written reflections on student teaching experiences at all three levels of reflectivity outlined by van Manen (1977): technical, practical, and critical. The extent of the student teachers' reflection at each level was influenced by their experimentation beyond covering the prescribed curriculum and the guidance they received during their reflection. Ginger's reflections on student teaching experiences were mainly at the technical level, secondly at the practical level, and to a much lesser extent at the critical level. On the other hand, Maryanne's reflections on student teaching experiences were mainly at the practical level and secondly at the critical and technical levels.

The cases of Maryanne and Ginger reveal the centrality of reflection in prospective mathematics teachers' development during student teaching. Mathematics student teachers' reflection interacts with their teacher knowledge, teaching, and supervision in the expansion of their teacher knowledge. As student teachers plan their lessons, they reason about their teaching, drawing on their teacher knowledge and enhancing it in relation to their level of reflectivity. As they implement their plans during teaching, they experiment with knowledge that supports their plans and expand that knowledge based on the implementation. Teaching provides an opportunity for them to reflect and develop their knowledge as they evaluate their lessons. Supervision, including support for experimentation, helps facilitate reflection beyond the technical level in both the planning and evaluation stages. When student teachers are restricted in their ability to experiment by external factors such as their supervising teacher or internal factors such as their understanding of the mathematics, the knowledge they draw on during planning is restricted and their level of reflectivity is hindered. Consequently, their teaching, reflection on teaching and thus their development of knowledge is compromised.

## Implications of this Study

This study provides implications for the education of prospective mathematics teachers and for continued research in the areas of teacher knowledge and reflection. The cases of Maryanne and Ginger accentuate the need to seek contexts that provide prospective mathematics teachers the opportunity to experiment with the knowledge and implement the cognitive processes (i.e., reflection) they are developing in their teacher education coursework. The cases reveal the need for prospective teachers to reflect beyond a technical level (with guidance as needed) in order to develop and understand the importance of teacher knowledge about students as learners of mathematics and about the teaching of mathematics that addresses the needs of these students.

A natural extension to this study would be a longitudinal study examining the development of teacher knowledge and reflection as teachers complete their teacher education program and then proceed in their teaching career. It would also be valuable to explore the role of reflection in field experiences other than student teaching and means for promoting reflection at the practical and critical levels within teacher education programs.

## References

Baird, J. R. (1992). Collaborative reflection, systematic inquiry, better teaching. In T. Russell & H. Munby (Eds.), *Teachers and teaching: From classroom to reflection* (pp. 33-48). New York: Falmer Press.

Brown, C. A., & Borko, H. (1992). Becoming a mathematics teacher. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 209-239). New York: Macmillan.

Carnegie Task Force on Teaching as a Profession. (1986). *A nation prepared: Teachers for the 21st century.* Washington, DC: Carnegie Forum on Education and the Economy.

Fennema, E., & Franke, M. L. (1992). Teacher's knowledge and its impact. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147-208). New York: Macmillan.

Grossman, P. L. (1990). *The making of a teacher: Teacher knowledge and teacher education.* New York: Teachers College Press.

Holmes Group. (1986). *Tomorrow's teachers: A report of the Holmes Group.* East Lansing, MI: Author.

Lanier, J. E., & Little, J. W. (1986). Research on teacher education. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (3rd. ed, pp. 527-569). New York: Macmillan.

Merriam, S. B. (1988). *Case study research in education.* San Francisco: Jossey-Bass.

Reynolds, M. C. (1989). *Knowledge base for the beginning teacher.* New York: Pergamon Press.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 15*(1), 1-22.

van Manen, M. (1977). Linking ways of knowing with ways of being practical. *Curriculum Inquiry, 6*, 205-228.

von Glasersfeld, E. (1984). An introduction to radical constructivism. In P. Watzlawick (Ed.), *The invented reality* (pp. 17-40). New York: Norton.

# USING VIDEOS TO PROVIDE "CASE-LIKE" EXPERIENCES IN AN ELEMENTARY MATHEMATICS METHODS COURSE

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This study was conducted to assess the impact of using videos to provide "case-like" experiences as part of the professional development program for prospective elementary teachers. Elementary teacher candidates in a Mathematics Methods course explored teaching and learning mathematics through the use of a number of different video tapes. The video tapes were selected to help portray what it means to teach and learn mathematics within the context of a standards-based reform environment. The videos were used to represent reality, to stimulate thought and debate, and, as teaching instruments, for study, examination, and discussion. One of the findings suggests that teacher candidates appeared to have evolved from didactic ideas about the teaching of mathematics toward a more student-centered philosophy by the end of the course. This paper discusses teacher candidates' responses to a "case-like" video analysis as these responses relate to this finding.

## Introduction

It is becoming increasingly apparent that learning to teach is a developmental process focused on understanding the dilemmas of teaching (Harrington, 1995). Preservice teacher preparation programs seek to provide experiences that support this process primarily through the clinical and field components. Many of the professional courses that precede the clinical and field components now also include field-based experiences. In addition, interest in and use of cases has been introduced as part of preservice programs to help prospective teachers focus on the dilemmas of teaching throughout their preparation (Harrington, 1995).

Merseth (1996) defines a case viewed from a teacher education perspective as:

> a descriptive research document based on a real-life situation or event. It attempts to convey a balanced, multidimensional representation of the context, participants, and reality of the situation. It is created explicitly for discussion and seeks to include sufficient detail and information to elicit active analysis and interpretation by users. (Merseth, 1996, p. 726)

Cases are developed to represent reality, to stimulate thought and debate, and, as teaching instruments, for study, examination, and discussion.

More recently, narrative case materials have become available that address dilemmas in teaching elementary and middle grades mathematics (Barnett et al., 1994; Schifter, 1996a,b). In addition, videos, as "case-like" materials, are now available for supporting reasoning about mathematics instruction and about students' thinking about mathematics (e.g., Kamii, 1990; Richardson, 1990; TERC, forthcoming; CGI; WGBH, 1995). While some research on the use of case-based pedagogy in general teacher education courses has been completed (e.g., Levine, 1995; Harrington, 1995), limited research has been conducted on what happens when cases are used as part of the professional development of prospective teachers with respect to teaching mathematics. In addition, little empirical evidence has been developed concerning the effects of video-based case pedagogy in teacher education (Copeland & Decker, 1966) and, by default, in mathematics teacher education.

## Purpose of the Study

The major question for this study was: What is the impact of using videos to provide "case-like" experiences in a mathematics methods course as part of the professional development program for prospective elementary teachers? The use of video tapes was not the only instructional strategy employed but was seen as central to accomplishing the goals of the course.

The use of a variety of video tapes was integrated into a course called Methods for Teaching Elementary School Mathematics (Mathematics Methods Course) using variations of the case methodology. The theoretical perspectives were developed through the use of readings (primarily from Van De Walle, 1994), explorations of the developmental sequence of the mathematics content over the K-6 curriculum, and a variety hands-on activities linked to teaching specific mathematics content.

The teacher candidates were in weekly field placements, that is, the classrooms in which they would student teach during the spring. However, it was not possible to rely on their encountering examples of topics and issues that were being addressed in the course in their placements. For this reason, video tapes were used to provide exemplars of communities (some better than others) in which making sense of mathematics was supported and developed so teacher candidates might practice analysis and contemplate action with respect to teaching mathematics. In addition, from mathematical autobiographies written at the beginning of the course, there was evidence that most of the teacher candidates viewed mathematics learning as "teaching by telling," and that several did not like and/or had had mixed experiences with mathematics. Consequently, the video tapes showing classroom lessons and student interviews also served as stimulants to personal reflection.

## Research Context

The Mathematics Methods course is part of a "methods block" taken by teacher candidates during the fall semester of their senior year. The section discussed had 19 teacher candidates (17 traditional age; 2 non-traditional age; 17 female/2 male; none with a second major in mathematics). The teacher candidates began the program during their junior year and completed courses focused on child development as well as a foundations/social studies methods course (each with a field-based component). With respect to mathematics, the teacher candidates met the general college requirement by selecting from a number of choices, several of which may not be relevant (e.g., Symbolic Logic) to teaching mathematics in the elementary grades.

As part of the classwork in the Mathematics Methods course, a number of different strategies were used to probe for responses to video "case-like" work. Initially, some of the WGBH (1995) tapes were used to focus teacher candidates' attention on the design and implementation of mathematics lessons, with the acknowledgment that, at this stage, their concerns were on management and classroom structure.

The next set of tapes was selected to accompany a section of the course that considered children's development of concepts related to number. These tapes (CGI; Richardson, 1990)were used as exemplars of the developmental issues and approaches to teaching about which teacher candidates were reading (Van de Walle, 1996) and of how to conduct interviews in preparation for their own assignment to interview students as part of the course requirements.

The third set of tapes (Kamii, 1990; TERC, forthcoming) accompanied a section of the course that considered the development of students' computational knowledge. The intent was to provide experiences in which teacher candidates saw a variety of children's reasoning strategies occurring in the context of whole class lessons.

The fourth set of tapes accompanied a section of the course that considered the development of concepts related to fractions. Each of two tapes was used as a "first lesson" that provided a particular interpretation of fractions and used a particular manipulative model.

A variety of data were collected. These included all course written work, journal writing tasks, and transcriptions of two sets of interviews conducted with teacher candidates during the course. Additional data were collected either during or immediately after the teacher candidates' field placements in spring, 1997 through interviews about the curriculum units they prepared as part of the Mathematics Methods course and the instructional sequences that occurred during the teaching of the units in the spring. As part of these interviews, the impact and value of the videos was discussed.

## Selected Findings

An initial analysis of the impact of using videos as part of the Mathematics Methods course has been reported elsewhere (Friel & Carboni, 1997). In this paper, teacher candidates' responses to a final examination task that involved an analysis of the video *Arrays and Fractions* (WGBH, 1995) are examined in light of one of the key findings included in the earlier report, that is, that teacher candidates appeared to have evolved from didactic ideas about the teaching of mathematics toward a more student-centered philosophy (Friel & Carboni, 1997).

As part of the final examination, teacher candidates were permitted to watch the video together and to brainstorm responses to a set of focus questions (See Figure 1); individuals each wrote their own responses to these questions. Since many of their prior experiences had been with videos that were used to discuss what was good about teaching, the use of this video tape was included to assess how teacher candidates might evaluate a lesson that involved more obvious dilemmas of instructional practice. Teacher candidates' responses to this video analysis provide further insights into their thinking with respect to moving toward a more student-centered philosophy.

The initial question in the set of focus questions involved clarifying the mathematical task addressed in the video. One teacher candidate briefly summarized the main points; the reader is encouraged to view this tape as only limited discussion of the content of the video is possible here.

> For this lesson (after an initial introduction and discussion of fractions and arrays), the children (grades 1-2) are given index cards with numbers and fractions on them. They are first to make an array showing the particular number using tiles and then show the fractional part of that number. For example, with the number "15" and the fraction "1/3", the child could first make a 3 by 5 array with tiles and draw it on grid paper. They could then circle or shade in five of the fifteen blocks of the array, illustrating that five is one third of fifteen.

Responses to two of the focus questions from the video analysis are discussed in this paper - teacher candidates reflections on the value of the task (part a) and their reflections about their continuing concerns with respect to teaching (part d).

While the majority of teacher candidates considered this as a potentially worthwhile mathematical task, 95% of the candidates raised concerns about the implementation of the task. For some (32%), their concerns focused on the nature of the task itself, including the apparent confusions about arrays and about fractions and the lack of connection built between the model (arrays) and the concept of fractions. Others (32%) felt that the students did not have enough background or were not developmentally ready to handle the concepts

Respond to the questions below. Provide justification and/or evidence for your responses, referencing resources used during the course.

a. Describe the mathematical task – the content of the lesson. Is it a worthwhile mathematical task? Explain your reasoning.

b. Describe the nature of the discourse that is taking place, considering the teacher's role and students' roles.

c. From your view from the "back of the room", comment on the overall lesson.

    - Were there any "missed or misjudged opportunities" to engage students or nurture student's mathematical thinking? If so, given an example and discuss.
    - What would you do for the next lesson based on what you observed in this lesson? Why?
    - Describe one way you might assess students' understanding of the content based on what you have observed. What is the purpose of this assessment?

d. What question(s) does this experience raise about teaching and learning mathematics?

**Figure 1** *Final examination video viewing question*

as presented in this lesson. Still others (36%) thought the potential for the overall lesson was "good;" its lack of success was tied to teacher behaviors that included limited wait time and limited attention to student responses in terms of understanding, developing, and building instruction based on student thinking.

The questions and concerns raised about teaching and learning mathematics from viewing this lesson provide evidence of a number of tensions that teacher candidates were experiencing by the end of the course. These include concerns about the role of the teacher and the place for explicit teaching of a concept, pacing of lessons and knowing when to "move on," which mathematics concepts and models are developmentally appropriate for children, designing lessons that are child-led and still accomplish the teacher's learning goals, judging when planned activities are too much "stuff" and not enough substance, and developing expectations about what students are capable of understanding. A few teacher candidates comments highlight some of these concerns.

> As I catch myself time and again stressing the importance of having students share their thinking and letting them explore rather than being the one to impart knowledge, I, too, wonder if there are certain learners who need more guidance.
>
> There are many ways to do things and show concepts. Which one is best? Which one helps the students most? How do you know if they are ready for a task? . . . I worry that I will not be able to develop interesting and stimulating lessons in mathematics. . . . I will not always be able to watch hundreds of videos to gain an idea for a few lessons. My hope is that one day I will be the one on the videos, helping other teachers-to-be learn. . . . Sometimes I wish I had a map and directions.

As one student noted, based on viewing this video, she now has a set of questions she will ask when preparing a lesson: what do students have to already know to succeed at the given lesson; do the activities in the lesson give everyone a fair chance to succeed; are the manipulatives or models selected appropriate to the task; and are the students familiar enough with the manipulatives or models to be able to construct their understandings of the topic at hand?

## Conclusion

The goal of the Mathematics Methods course was to consider what it means for elementary students to learn (and teachers to teach) mathematics through the use of challenging problems which may be explored collaboratively and through class discussion of students' solutions. Video tapes in case-like problem situations were used as a vehicle to promote such understanding. It was found that teacher candidates appeared to evolve from didactic ideas about the teaching of mathematics toward a more student-centered philosophy. Responses to questions that focused teacher candidates' discussion of the *Arrays to Fractions* video (WGBH, 1995) provide additional insights into what it means for these teacher candidates to move to a more student-centered philosophy. In discussing the lesson, teacher candidates identified dilemmas and associated tensions that focused on the purposes of the task, the presentation of the task, and the task as carried out by students that provide further evidence for this shift in thinking about teaching.

## References

Barnett, C., Goldenstein, D. & Jackson, B. (Eds.), (1994). *Fractions, decimals, ratios, and percents: Hard to teach and hard to learn?* Portsmouth, NH: Heinemann, Inc.

Copeland, W. D., & Decker, D. L. (1996). Video cases and the development of meaning making in preservice teachers. *Teaching and Teacher Education. 12*(5), 467-481.

Corwin, R. (1996). *Talking mathematics: Resources for developing professionals.* Portsmouth, NH: Heinemann, Inc. Videotape: Classroom Episodes.

(CGI) *Cognitively guided instruction.* Madison, WI: Wisconsin Center for Educational Research, School of Education, University of Wisconsin. Set of video tapes. (Contact E. Fennema)

Friel, S. N., & Carboni, L. W. (1997, February). *Making sense of teaching and learning mathematics: Using videos to provide "case-like" experiences in an elementary mathematics methods course.* Paper presented at the Annual Meeting of the Association of Mathematics Teacher Educators, Washington, DC.

Harrington, H. L. (1995). Fostering reasoned decisions: Case-based pedagogy and the professional development of teachers. *Teaching and Teacher Education, 11*(3), 203-214.

Kamii, C. (1990). *Multiplication of two-digit numbers: Two teachers using Piaget's theory.* New York: Teachers College Press. Videotape.

Levine, B. B. (1995). Using the case method in teacher education: The role of discussion and experience in teachers' thinking about cases. *Teaching and Teacher Education, 11*(1), 63-79.

Merseth, K. K. (1996). Cases and case methods in teacher education. In J. Sikula, T. J. Buttery, & E. Guyton (Eds.). *Handbook of research on teacher education* (pp. 722-744). New York: Macmillan.

Richardson, K. (1990). *A look at children's thinking* . Norman, OK: Education Enrichment, Inc. Videotape and study guide.

Schifter, D. (1996a). *What's happening in the mathematics classroom: Issues of practice in teacher narratives from the mathematics reform movement (Vol. 1).* New York: Teachers College Press.

Schifter, D. (1996b). *Constructing a new practice: Issues of practice in teacher narratives from the mathematics reform movement (Vol. 2).* New York: Teachers College Press.

TERC (Forthcoming). *Developing mathematical ideas: building a system of tens/making meaning of operations* . Cambridge, MA: Author. Videotape.

WGBH Boston. (1995). *Teaching math: A video library, K-4.* South Burlington, VT: The Annenberg/CPB Math and Science Collection. Videotapes and study guides.

Van De Walle, J. A. (1994). *Elementary school mathematics (2nd ed.).* White Plains, NY: Longman Publishing.

# MATHEMATICS CULTURE CLASH: NEGOTIATING NEW CLASSROOM NORMS WITH PROSPECTIVE TEACHERS

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As part of a 3-year study of their mathematical and pedagogical development, the second author taught a mathematics course for prospective elementary teachers. His attempts to engage these traditionally-educated undergraduates in inquiry mathematics brought about what we came to regard as a *culture clash*. Our analysis suggests that by the latter stages of the course, a classroom microculture characterized by inquiry mathematics had evolved. We examine the processes by which the participants in this classroom community negotiated norms and practices that supported their mathematical activity. We identify four categories of interaction central to the ongoing negotiation, and illustrate how each contributed to the negotiation of new norms and practices and the mediation of the culture clash.

The practice of elementary mathematics teaching is deeply interconnected with teachers' theories of teaching, learning, and mathematics. For prospective teachers, opportunities to reconstruct these personal theories can result from their participation in a mathematics classroom community that is characterized by the mathematical and social practices of inquiry mathematics (Richards, 1991). Mathematics classes in teacher preparation programs therefore have great potential to alter prospective teachers' relationships to mathematics as well as to mathematics learning and teaching (Simon, 1994).

There are several examples of mathematics classrooms characterized by inquiry in which students and teacher engage in new (to school mathematics) forms of activity. Cobb, Yackel, and Wood (1989) focus particularly on how one teacher and her students renegotiated their classroom practices in response to situations in their second grade mathematics class. We build on and extend this work to examine the process by which new norms and practices are negotiated in a mathematics course for prospective elementary teachers in which the instructor endeavored to engage traditionally-educated undergraduates in inquiry mathematics. We highlight the tension between the mathematics cultures of teacher and students (in this case, prospective elementary teachers) and thus the negotiative nature of the constitution of a classroom mathematics culture. We identify particular kinds of interaction through which this negotiation process took place. The longer paper provides a more detailed discussion.

## Theoretical Perspectives

Drawing on the work of Bauersfeld, Krummheuer, and Voigt (1985), we see human interaction as a process of ongoing and evolving interpretations of each other's actions and responses to those actions, and use Voigt's (1985) term *negotiation* to describe these processes. Negotiation of meaning in this sense occurs implicitly through the subtle adaptation of participants' actions to fit with their ongoing interpretations of each other's words and actions. This negotiation can also be explicit and easily observable.

A classroom microculture is therefore neither static nor created by the teacher (Bauersfeld, 1996), but is continually reconstituted through the mutual adaptation of its participants' interpretations of activity. As they gradually come to understand each other, participants' interpretations become coordinated into routine responses, and patterns of interaction emerge. Individual sense-making and the classroom microculture are thus reflexively related; individuals contribute to shaping the microculture, while the microculture enables and constrains the mathematical activity of the individual participants (Cobb & Bauersfeld, 1995). Thus, changes in a classroom microculture occur through evolution rather than rearrangement (Voigt, 1995).

Where teacher and students bring compatible expectations and interpretations to classroom events, the microculture is constituted and reconstituted smoothly. It might appear to participants, as well as to the casual observer, that no negotiation occurs at all. However, where classroom participants bring different expectations and interpretations, there are instances of misunderstanding and miscommunication, and moments where previously implicit understandings must be made explicit. Such "cultural" differences are inherent in classes in which a teacher educator representing inquiry mathematics works with a group of prospective teachers who are steeped in the culture of school mathematics. It is this situation that we characterize as a *culture clash.*

## Data and Analytic Technique

This study focuses on the first course of the Construction of Elementary Mathematics Project, a 3-year investigation of the mathematical and pedagogical development of elementary teachers. Simon, the second author, taught this mathematics course, focusing on multiplicative structures, to a group of prospective teachers in the context of a whole-class teaching experiment (Simon & Blume, 1994). Our primary sources of data were videotapes and transcriptions of the 25 class meetings of this semester-long course. We also examined copies of the students' weekly journal reflections and written work. Because students' self-report data could have been biased by what they perceived to be the "party-line," we considered students' written and oral assertions significant to the analysis only when their assertions conflicted with teacher intentions. Thus, we inferred the students' points of view primarily from their participation in class activities.

Using analytic techniques of Bauersfeld, Krummheuer & Voigt (1985), we examined the class-room transcripts line by line for meanings the participants seemed to attribute to each other's words and actions. We worked our way through the transcripts of the entire semester (Classes #1-25) documenting the community's mathematical and social practices at each point in time. We tried to discern the community's definitions of teacher and student roles by comparing our understanding of the students' perceptions of their role to the teacher's expectations of students and, reciprocally, we compared our understanding of the students' expectations of the teacher to his expectations of himself. We also analyzed the community's definitions of legitimate mathematical activity and what constituted acceptable justifications and explanations. Following Cobb, Wood, Yackel, & McNeal (1992), we consider an explanation to be a statement intended to clarify a communication, and a justification to be a statement intended to convince someone of the validity of an idea.

Our initial analysis of videotapes for the entire semester showed students engaging in discussions that were remarkable for their mathematical substance and for the non-traditional teacher and student roles as early as Class #15. We thus took this to be indicative of changes that had occurred since the first class, and examined the specific norms and practices that were evident in Class #15. We then reexamined Classes #1-15 exploring how this group learned to interact in such a way as to produce the discussions exemplified in Class #15. We looked across transcripts for categories of events and interactions that appeared to have contributed to the development of the observed social and mathematical practices. We also examined events that uncovered participants' differing assumptions, such as instances of overt disagreement or miscommunication and used these discrepant events to elaborate on our interpretations of students' points of view. The resulting analysis attempts to explain both apparent patterns and discrepancies (Erickson, 1986).

## Negotiating Classroom Norms

New classroom norms and roles for teacher and students were negotiated through interactions involving new experiences, the journal, paradigm cases, and discrepant events.

### Interactions Involving New Experiences for Students

In Class #2, teacher and students began trying to communicate across their different mathematical cultures. The process of negotiating new practices and new understandings of old practices began with some explicit comments from the teacher about what he wanted them to be doing during group work and whole class discussions (e.g., explaining their thinking, trying to understand each other's thinking) and what he would be doing (e.g., asking for a paraphrase, monitoring wait time). Such explicit statements seemed necessary given the teacher's anticipation that his expectations would differ significantly from the students' prior experiences, but they were not sufficient to establish such practices as normative.

Traditionally-educated students are not in the habit of offering justifications. So, in order to engage his students in mathematical inquiry, the teacher tried to promote an atmosphere in which both teacher and students ask "Why," and students *expect* to justify their claims. For example, when students said they had multiplied to find the number of copies of a cardboard rectangle that would cover a table, the teacher asked them why this was a good way of going about the problem.

De: Because, in previous math classes, you learned the formula for area is length times width so probably everybody has the idea.

T: And all those evil math teachers you were talking about before (laughter) and you're gonna now take their word for it?

Mo: They showed us that.

T: How do we know if they are right?

Mo: Because they showed us. (laughter)

T: Blind faith?

Mo: The teachers, you know, they showed us how it worked.

Although they did not speak directly about it, Class #2 began the negotiation of the practice of justification. Rather than simply telling students that he would only accept statements that they could justify, the teacher attempted to have the students experience a need for mathematical justification and subsequently a need for more sophisticated forms of justification. Over time, they grew accustomed to this practice and offered increasingly sophisticated justifications. Simon & Blume (1996) discuss the evolution of different levels of justification in this classroom community.

Through such interactions, students seemed to form the impression that the teacher might indeed ask "why" in response to any statement made in class. Some of the students' journal responses reflected frustration with their inability to satisfy the teacher, as perceived from his continued questioning. At this time, students probably did not feel a need to convince themselves, rather they perceived a need to figure out what the teacher wanted. We saw the events in Class #2 as the meeting of different mathematics cultures. Such new experiences provided students with an opportunity to re-interpret their assumptions about student and teacher roles in mathematics class.

### Interactions Involving the Journal

After each class, the teacher read and wrote comments on students' journal responses to that class. He routinely began the next class with comments to the group about issues arising in the journals that he considered relevant to the whole group. We came to view this teacher talk about journal themes as a part of the process of negotiating mathematical and social practices appropriate for competent participation in this community: Students participate in an activity in class; students write a journal entry; the teacher responds individually and to the group; students adapt their classroom activities and journal writing in response to his comments; the teacher comments again in response to students' activities and writings.

Interactions involving the journal contributed to the negotiation process by raising issues of affect, uncovering differing interpretations of class activities, and clarifying the purpose of the journal. Specifically, the teacher's public responses addressed students' feelings about this mathematics class, students' and teacher's roles, the goals of the course, student understanding of the mathematical content, and the importance and meaning of articulating mathematical ideas. There were also occasions on which the teacher responded to journal entries with action rather than words, for example, by posing a problem or initiating a discussion.

### Interactions Involving a Paradigm Case

In Class #4, the students spontaneously initiated a mathematical exploration for the first time. The fortuitous emergence of the kind of mathematical activity that he was seeking to engender offered the teacher an opportunity to make his expectations concrete by pointing to a shared experience. In Class #5, he framed the events of Class #4 as a *paradigm case* (Cobb, Yackel, & Wood, 1989) of appropriate mathematical activity by highlighting specific behaviors. In this way, he prompted and guided conscious reflection on and interpretation of their prior activity and his response to their activity. Although experience itself has a notable effect on community members, explicit highlighting of a paradigm case can be an even more powerful means of affecting classroom norms and practices because it brings together implicit understandings with officially approved interpretations.

### Interactions Involving a Discrepant Event

As long as interaction proceeds smoothly, participants tend to assume shared meanings, norms, and practices. Periodically, however, an event occurs in the life of a classroom microculture that exposes a lack of fit among the perspectives of the participants. Announcement of the first exam was such an event. The resulting discussion demonstrated that although community members' understandings of teacher and student roles had become more compatible, significant differences in their perceptions of these roles remained.

Having experienced non-judgmental responses to their classroom contributions, students seemed to have constructed a belief that effective participation consisted of honest expression of ideas and their rationale. Thus, they did not see the particular mathematical content of their ideas as a key factor in effective participation. The notion of an exam that would be used to evaluate and grade their understandings was seen as counter to the evolving microculture of this mathematics class. One student asked how the teacher could grade their understanding, "that's kind of like our opinion." Another student asked if a good argument was good enough to be judged correct, or in reverse, if a good argument with an incorrect conclusion was going to be marked wrong. The students' notions of teacher had shifted from a view of teacher as "one who imparts knowledge and determines validity" toward teacher as "one who poses problems and facilitates discussion without evaluative judgment."

## Conclusions

The analogy of a culture clash illuminated the mutually adaptive process by which this group constructed norms and practices that supported mathematical inquiry. By viewing the teacher's and students' differing perspectives (including expectations of each other) as "culturally"-based, we have been able to understand how deeply ingrained these perspectives and roles are, and how slow and complex the process of change. This analogy also served to highlight the interconnection between mathematical and social understandings; challenging students' mathematical ideas and justifications not only provoked cognitive reorganization in many students, but also reinterpretation of their role in mathematics class. As teacher educators attempt to engage their students in new forms of mathematical activity, there is increased potential for such culture clashes. Because we believe that the negotiation of new norms and practices for school math will play a key role in the ultimate success of reform efforts, we advocate continued study of these culture clashes.

## References

Bauersfeld, H. (1996). *Social constructivism, classroom cultures, and other concepts—What can they mean for teachers?* Plenary address of the Eighteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Panama City, FL.

Bauersfeld, H., Krummheuer, G., & Voigt, J. (1985). Interactional theory of learning and teaching mathematics and related microethnographical studies. In H. G. Steiner & A. Vermandel (Eds.), *Foundations and methodology of the discipline of mathematics education* (pp. 174-188). Antwerp: Proceedings of TME Conference.

Cobb, P., & Bauersfeld, H (Eds.). (1995). *The emergence of mathematical meaning: Interaction in classroom cultures.* Hillsdale, NJ: Lawrence Erlbaum.

Cobb, P., Wood, T., Yackel, E., & McNeal, B. (1992). Characteristics of classroom mathematics traditions: An interactional analysis. *American Educational Research Journal, 29*(3), 573-604.

Cobb, P., Yackel, E., & Wood, T. (1989). Young children's emotional acts while doing mathematical problem solving. In D. B. McLeod & V. M. Adams (Eds.), *Affect and mathematical problem solving: A new perspective* (pp. 117-148). New York: Springer-Verlag.

Erickson, F. (1986). Qualitative methods in research on teaching. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (3rd ed., pp. 119-161). New York: Macmillan.

Richards, J. (1991). Mathematical discussions. In E. von Glasersfeld (Ed.), *Radical constructivism in mathematics education* (pp. 13-52). Dordrecht, The Netherlands: Kluwer.

Simon, M. (1994). Learning mathematics and learning to teach: Learning cycles in mathematics teacher education. *Educational Studies in Mathematics, 26,* 71-94.

Simon, M., & Blume, G. (1996). Justification in the mathematics classroom: A study of prospective elementary teachers. *Journal of Mathematical Behavior, 15*(1), 3-31

Simon, M., & Blume, G. (1994). Mathematical modeling as a component of understanding ratio-as-measure: A study of prospective elementary teachers. *Journal of Mathematical Behavior, 13,* 183-197.

Voigt, J. (1985). Patterns and routines in classroom interaction. *Recherches en Didactiques des Mathematiques, 6*(1), 69-118.

Voigt, J. (1995). Thematic patterns of interaction and sociomathematical norms. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 163-201). Hillsdale, NJ: Lawrence Erlbaum.

# A GROWTH OF KNOWLEDGE OF A MIDDLE SCHOOL TEACHER

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Evidence suggests that middle-grade teachers do not have the required mathematical, pedagogical content, and psychological knowledge. This research describes a case study focusing on the changes made by one teacher in these three areas.

The teacher is one of thirty-five middle school teachers who are participating in a two-year NSF teacher enhancement program. Teachers were required to commit to a day-long, four-week inservice during the summers of 1996 and 1997, and attend five inservice meetings during the two intervening academic years. The summer institutes reflected the integration of mathematical, pedagogical content, and psychological knowledge and its role in effective teaching. The underlying theoretical orientation of the program and the way it was conducted reflected a social constructivist view which assisted teachers in constructing their own knowledge. A sixth- through eighth- grade teacher, Wendy, who participated in this program is the subject of this research. The longitudinal case study was developed in a period of one year from seven primary sources: field notes on observations of the teachers during summer institute and five one-day workshops; journal reflections documenting her learning of specific mathematics and mathematical connections she made; pre- and post-instructional assessments on her knowledge of algebra, number, probability, geometry, measurement, and statistics; interview with the teacher about specific mathematical connections she made; teachers' belief scale about the nature of mathematics learning and teaching; and classroom observations of the teacher teaching (27 lessons) over four months.

Wendy had a less than adequate mathematical prerequisite and was anxious about her mathematics understanding. She developed in her mathematics knowledge through group activities and discussion and became more confident and more comfortable with solving mathematical problems. Wendy willingly considered the philosophy and methods that were being discussed and suggested in the enhancement program, and effectively implemented them in her classroom. For example, she emphasized students' conceptual rather than procedural understanding during her fraction lessons over one month (Cramer et al. 1997). Her questions demonstrated her view of the importance of students' reflecting on their own thinking, reasoning, and communication. Her level and sophisticated progressed over the course of this study while she continued to believe in the utility of mathematics. This session will discuss the types of growth which Wendy expressed.

## References

Cramer, K., Behr, M., Post, T., & Lesh, R. (1997). *Rational Number Project—Fraction lessons for the middle grades level 1.* Dubuque, IA: Kendall/Hunt.

# COGNITIVE AND AFFECTIVE PREDICTORS OF PRESERVICE TEACHER EFFECTIVENESS IN ELEMENTARY MATHEMATICS PEDAGOGY

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The purpose of the typical college level elementary mathematics methods course is to combine mathematical content and instructional strategies into a resultant collection of "pedagogical content knowledge" specific to the teaching of mathematics to elementary school children, while at the same time setting them at ease with respect to their attitudes towards mathematics, and also possibly improving their spatial skills. University courses designed with this intent range from those with a textbook orientation, to those with a hands-on laboratory approach, some including field practicums. There is a continuing debate among mathematics educators as to which of these areas should be stressed in the preparation of elementary teachers (content and concept knowledge, methodology, field experiences, attitude, spatial skills, etc.). The present study was designed to link some of these variables together in an attempt to build a theoretical model for the prediction of effective preparation of prospective teachers of elementary school mathematics.

All individuals in two classes of preservice elementary teachers (N = 48) were video taped as they taught three different self-designed teaching episodes to their peers on three different occasions. The three episodes were focused on subtraction with regrouping, multiplication of fractions, and area/volume, and were designed to be delivered to elementary school children. The 144 tapes were independently evaluated by three experienced elementary mathematics methods instructors, according to established criteria, and were assigned numerical ratings in three areas: evidence of content knowledge, appropriateness of methodology, and effectiveness of delivery. These video tapes were devised as a plan to capture the actual effectiveness of the students when they synthesized and applied what they had acquired from the course. The other four variables of quiz average, mathematics skills score (50 item inventory), mathematics anxiety score (Revised Mathematics Anxiety Rating Scale [RMARS]), and spatial ability (spatial section of the Differential Aptitudes Test [DAT]) were then investigated as potential predictors of teacher effectiveness. Tables of correlations and multiple regressions were computed, with data analyses carried out for total video rating scores, and also for individual content, methodology, and delivery ratings.

Of the four potential predictor variables, quiz score was the only significant predictor of preservice teacher performance on the video tapes of teaching episodes. It accounted for approximately 16% of the variance in preservice teacher performance. The other three variables of basic math skills, math anxiety, and spatial skills were significantly correlated with each other, but were not correlated to quiz scores or teacher performance. However, when added to the regression, both basic skills score and math anxiety increased the variance accounted for in teacher performance to approximately 25%. This indicates that there is common variance shared between each of these two variables and the predictor variable "quiz score" which is irrelevant to the prediction of video taped teaching performance, and thus when these factors are partialed out, the prediction model is improved.

The implication of this study is that pure mathematics skills (content knowledge), the affective aspects of mathematics learning (mathematics anxiety), and spatial skills do not impact preservice pedagogical effectiveness (as measured by video taped performance assigned in a methodology course) as compared to course quiz grades designed as measures of acquisition of specific teaching skills for mathematics (pedagogical content knowledge). This implies that mathematics methodology courses for preservice elementary teachers can be successful for most students, regardless of their prior levels of basic math skills, spatial skills, or fears concerning mathematics. This is perhaps a somewhat positive finding for instructors of elementary mathematics methods courses.

# GOALS FOR UNDERSTANDING MATHEMATICS IN CONTENT COURSES FOR PROSPECTIVE ELEMENTARY SCHOOL TEACHERS

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Historically, investigations into understanding have focused on student understanding and, more recently, teacher understanding of student understanding. My purpose is to push this further, asking us to consider what we can learn from the study of university teachers' understanding of mathematics and its study. In presenting this report, I provide an analysis of research results around which thoughtful mathematics educators can discuss goals and the means to those goals in the mathematical preparation of teachers, particularly elementary school teachers. I consider how the thoughts and actions of teacher educators affect the kind of mathematics taught and students' opportunities to learn mathematics.

The study from which this report arose focused on the roles mathematics instructors saw as important for mathematics content courses to play in elementary teacher preparation programs. A deeper analysis of the most valued role, that of "learning mathematical content," focused on beliefs about knowledge and understanding from the perspectives of the teacher educators. My analysis led to the articulation of two distinctive goals for understanding of the mathematics content. The goals are supported by particular patterns of thought and action on the part of mathematics instructors. I will lay out the characteristics of the two goals for understanding and consider the implications of each goal on teacher decision-making and the potential of each goal for enhancing student learning.

# EXAMINING ELEMENTARY TEACHER BELIEFS ABOUT MATHEMATICS, MATHEMATICS TEACHING, AND TECHNOLOGY

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This study describes the beliefs of two elementary teachers who were participants in a five year teacher-enhancement project focused on technology. Survey data were collected at both the beginning and end of the five year period; interview data were collected at the end of the project. The purpose of this study was to describe teachers' beliefs about mathematics, mathematics teaching, and technology. Little research on teacher beliefs about technology use in the elementary mathematics classroom has been conducted. This study builds a theoretical framework for categorizing elementary teacher beliefs about technology use in instruction with regard to their beliefs about mathematics and mathematics teaching.

Participants were described as having empiricist, trivial constructivist, or social constructivist beliefs about mathematics. Their beliefs about teaching were described by four dominant views of how mathematics should be taught (Kuhs & Ball, 1986). Studies describing technology use in mathematics (Fleener, 1994; Lampert, 1988) so far have provided data that suggest two categories of beliefs. Teachers with *Mastery* beliefs hold that technology should not be used until mathematical concepts are mastered. Teachers who possess *Exploratory* beliefs use technology to introduce concepts. One case fell in neither of these categories. Therefore a third category, which falls between the first two, is proposed. Teachers in the *Pre-Mastery* category, believe that students should be introduced to a mathematical concept without technology, but can begin to use technology even if the concept has not yet been mastered. Technology provides a tool for students to further explore concepts and reinforce ideas they have already encountered. The table below summarizes expected patterns of mathematics, teaching, and technology beliefs.

# AWARENESS OF THE STANDARDS, MATHEMATICS BELIEFS, AND CLASSROOM PRACTICES OF MATHEMATICS FACULTY AT THE COLLEGIATE LEVEL

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Beginning with the 1989 publication of the *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics [NCTM]), there have been many calls for change in how mathematics is taught (e.g., National Research Council, 1989; Steen, 1989). The three goals of this investigation of mathematics teaching at the collegiate level were: 1) to gather information from collegiate mathematics faculty on their beliefs related to mathematics, mathematics learning, and mathematics teaching; 2) to determine the extent of their knowledge or awareness of ongoing reform efforts, in particular their awareness of the NCTM *Standards* documents (1989, 1991, 1995); and 3) to investigate the classroom teaching practices they used. Twenty-six collegiate mathematics faculty at seven institutions were interviewed. Results indicate that the faculty had low levels of awareness of current reform efforts generally, and of the NCTM *Standards* specifically. To the extent that they were aware of the *Standards*, many had favorable impressions of the recommendations included in these documents. Moreover, many of their stated mathematics related beliefs coincided with the underlying assumptions of the *Standards*. However, this appeared not to have transferred to their teaching practices; the data collected relative to classroom teaching practices indicate that the traditional mode of instruction—students taking notes while the teacher lectures—was still the most frequently used method at the collegiate level.

## References

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: The Council.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: The Council.

National Council of Teachers of Mathematics. (1995). *Assessment standards for school mathematics*. Reston, VA: The Council.

National Research Council. (1989). *Everybody counts: A report to the nation on the future of mathematics education*. Washington, DC: National Academy Press.

Steen, L. A. (Ed.). (1989). *Reshaping college mathematics* (MAA Notes 13). Washington, DC: The Mathematical Association of America.

# INCREASING PROSPECTIVE SECONDARY SCHOOL MATHEMATICS TEACHERS' UNDERSTANDING OF THE COMPLEXITY OF TEACHING

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Current reforms in mathematics education have increased the demands placed on classroom teachers by shifting the definition of good teaching from "well-planned and executed presentations" to "carefully chosen tasks and appropriate questioning." To prepare secondary school mathematics teachers to be successful in a reform environment, mathematics methods courses must address the complexity of teaching. This study investigated secondary mathematics methods students' responses to the complexity of a classroom situation as described in a written case.

Research based on Cognitive Flexibility Theory (e.g., Spiro, Coulson, Feltovich, & Anderson, 1988) suggests that the best way to encourage transfer of knowledge in an ill-structured domain, such as teaching, is to consider the domain as a landscape to be "criss-crossed." Cases were used as a pedagogical approach in the methods course because of their ability to "criss-cross" the domain of mathematics teaching. The particular case focused on in this study involved an algebra experiment designed to strengthen student understanding of linear functions. The case describes segments of what went on in the high school algebra class and the teacher's reactions.

The methods students' written responses to case questions, the video-taped case discussion, and follow-up via an electronic conference were analyzed to assess the students' awareness of the complexity of teaching. Three areas were specifically looked at in this study: 1) questioning techniques, 2) placement of investigative activities in the curriculum, and 3) assessing student understanding. Some students grasped the complexity of classroom mathematics instruction from reading the case. Others required the case discussion to see that even seemingly minor actions by the teacher could have significant impact on the nature of learning that occurred in the classroom. The presentation will look at the nature of the prospective teachers' understandings.

## Reference

Spiro, R. J., Coulson, R. L., Feltovich, P. J., & Anderson, D. K. (1988). *Cognitive flexibility theory: Advanced knowledge acquisition in ill-structured domains* (Technical Report No. 441). Champaign, IL: University of Illinois at Urbana-Champaign.

# PRE-SERVICE TEACHERS' VALIDATIONS OF MATHEMATICAL SOLUTIONS

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Informal observation in previous classes convinced us that many of the students who entered our mathematics content course for preservice elementary teachers held a view of mathematics that was rigid and algorithmic. They lacked feelings of autonomy and confidence in their own mathematical ability and required external validation of their mathematical processes. We felt that teachers who held a rigid, non-autonomous view of mathematics could not be expected to support the reform effort that we consider critical to improve mathematics education. This project allowed us to attempt to begin to examine preservice teachers' views and confidence in their mathematical ability.

## Perspectives(s) or Theoretical Framework

Our classes are designed to fit closely with Dewey's and Piaget's ideas that learning mathematics is a constructive process and to reflect our philosophy that learning is neither wholly subjective nor totally framed in social interactions. Lave (1991) elegantly provides us with words upon which we cannot improve:

> Learning is recognized as a social phenomenon constituted in the experienced, lived-in world, through legitimate peripheral participation in ongoing social practice; the process of changing knowledgeable skill is subsumed in processes of changing identity in and through membership in a community of practitioners; and mastery is an organizational, relational characteristic of communities of practice. (p. 64)

The problem we chose to examine was situated in a context that was familiar to our students–taking brownies to a school bake sale. During the eight interviews, students discussed their problem solving process and their feelings of confidence about the accuracy of their solutions.

## Evidence and Results

A critical result of this study was evidence that preservice teachers did not use mathematics to justify their processes. Previously we had thought the learning of mathematics served to justify or explain the real world, and the real world allowed students to make the mathematics that they learned meaningful. However, for these students, truth was not in the accuracy of the mathematical model but was in the real world context. We had thought that the connection to the real world meant that students could validate the mathematics in the sense that the mathematics became useful or *valid*. However, we didn't understand that students used the real world to validate mathematics in the sense that the accuracy of the mathematics was being judged through the real world setting. These students did not feel confident when they based their justifications in their mathematical model. Students who relied upon their choice of mathematical model to validate their solution were, in general, not confident of that solution but instead reiterated that they didn't think that their solution was correct.

We found that truth is not in the axiomatic structure of the mathematical system, but rather the validity of the mathematical process is located in the real world. Reform in mathematics education depends on change occurring at the earliest level of a child's formal schooling. As we gain insight into how preservice elementary teachers view and use mathematics, we are better able to construct learning activities which broaden their conceptual basis.

## References

Lave, J. (1991). Situated learning communities of practice. In L. B. Resnick, J. M. Levine, & S. D. Teasley (Eds.), *Perspectives on socially shared cognition* (pp. 63–82). Washington, DC: American Psychological Association.

# USING HISTORY TO FURTHER THE UNDERSTANDING OF MATHEMATICAL CONCEPTS

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This study addresses the following question: In what ways are students better able to understand a difficult mathematics concept or idea by studying the historical development of the concept or idea?

Sfard (1994) compared the historical development of algebra with individual students' development of algebraic ideas. I investigated this with students in a History of Math class. After writing a short essay identifying some mathematical concept that had caused them difficulty, and discussing the historical roots of this concept, students were then asked to develop a major 10-12 page paper focusing on the historical development of the topic, which identified problems encountered historically, such as the resistance of the mathematics community to accept the idea, and identifying ways in which such difficulties were overcome. In conclusion, they were asked to discuss whether their understanding of the concept had improved as a result of the investigation.

Students selected a wide range of mathematical topics to write on, from the idea of what constitutes a mathematical proof and different proof techniques, to the development of the concept of limit, continuity, mathematical proof, infinite sum, derivative and integral. All students reported learning much from following the historical development of their chosen concept.

## Conclusions

1. Students reported a firmer grasp of the concept through the study of its historical development. It was encouraging for them to realize that many well-known mathematicians (Gauss, Cauchy, Fermat, Newton to name a few) had struggled with the very ideas that had given them trouble.
2. They were surprised to note that many concepts, integral and differential calculus for example, had taken many hundreds of years to develop fully. For example, one student wrote about the different definitions of continuity in use before the mathematics community settled on one.
3. Students discussed the "wider perspective" gained from writing such a paper, and wrote that they now felt it was more acceptable to them to have difficulty with a mathematical concept. They understood that it was acceptable to have a gradual or partial understanding of a concept,

realizing that it takes time to solidify a deep understanding of such topics.

4. Students reported that they now understood the changing nature of mathematics and the nature of the presentation of ideas. This helped them realize that mathematics is an ongoing field of study, rather than a static one. For example, several students wrote on the nature of mathematical proof, and were surprised to learn that this idea changed over time.

It seems clear that students have much to gain in researching the historical development of a mathematical concept, particularly one which has given them difficulty. This assignment focused the students and the resulting papers showed a depth not usually seen in math history papers, which often tend to be solely biographical and/or chronological.

## References

Sfard, A., & Linchevsky, L. (1994). The gains and pitfalls of reification—the case of algebra. *Educational Studies in Mathematics, 26*, 191-2.

# REPLICATING COMBINATORICS RESEARCH : THE EXPERIENCE OF PRESERVICE TEACHER "RESEARCHERS"

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Replicating tasks from appropriate research projects is an effective way to provide preservice teachers with a quality field experience within the time constraints of a mathematics education course. After their own investigations with combinatoric tasks and reading English's study (1992, 1996), students in the Statistics and Probability for Elementary Teachers course of Fall, 1996, replicated English's study with a class of 4th graders. Conditions in which these preservice teachers had to collect their data were far from ideal, but reasonable given the circumstances. As in the English study, fourth graders try to find all the different outfits for a bear by placing concrete objects on bear images as preservice "researchers" recorded their choices in order. Later, students give written responses to three parallel questions.

On the basis of the order and nature of combinations for each task, a child's strategy is coded as one of English's stages: nonplanning, transitional or odometer. The experience of collecting this data, identifying the strategies, and writing analyses of students is clearly a valuable learning experience for preservice teachers. The same is true for their experience with student diversity—differences across students and tasks—as it helps to highlight the importance of anticipating and handling diversity in the classroom. Diversity is also apparent in the way these future teachers pose the tasks to the children, code their combinations and strategies, and think about the mathematics and the children's performance. While such variations are interesting and useful in assessing knowledge and potential of these potential teachers, they also have the potential for introducing some distortion to the data and the findings of the authors.

McKinley, Shunta, and Shroyer analyzed the data provided by the class The story of what they find and learn reveals the benefit of going beyond data collecting and reporting levels to assume the more extensive and responsible role of researcher. Findings based on the 4th grade students include:

1. McKinley and Shunta believe that English's transitional stage, the one at which most students seem to function, does not reflect the qualitative differences evident within this stage. As a consequence they introduce two subcategories and use them to recode the student data.

2. Appropriate display of data allows for tracing the progress of individuals from item to item and across parallel items. Some interesting pat-

terns emerge, but to be significant, they need further verification and support.

3. Learning experiences for McKinley and Shunta, the preservice members of the team, demonstrate movement to higher levels of understanding.

Details on all of these findings are contained in the poster presentation.

## References

English, L. (1996). Children's construction of mathematical knowledge in solving novel isomorphic problems in concrete and written form. *Journal of Mathematical Behavior, 15*(1), 81-112.

English, L. (1992). Problem solving with combinations. *Arithmetic Teacher, 40*(2), 72-77.

# CHALLENGES OF REFORM IN TEACHER PREPARATION: ESTABLISHING AN AGENDA FOR DOCUMENTING MEANINGFUL CHANGE THROUGH RESEARCH

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The recent Third International Mathematics and Science Study (TIMSS) reflects a curriculum in the United States that is "a mile wide and an inch deep" (Schmidt, 1996). A change is necessary; this discussion session focuses on establishing an agenda for research on the critical components of the reform ideas in teacher preparation.

At the university level, the learning of and about mathematics occurs in two distinct sets of courses, content courses and methods courses. Clearly, a continuing dialogue between mathematicians and mathematics educators is needed to encourage collaboration and cooperation in the process of the teaching and learning of mathematics. The courses required for pre-service teachers must help instill the vision reflected in recent National Council of Teachers of Mathematics (NCTM) documents. Fisher and Leitzel (1996) provide guidelines for reform specific to teacher preparation. These guidelines suggest directions for needed research.

Participants will contribute through a) an introductory brainstorming activity to collect an extensive list of research questions and/or topics that connect reform issues in mathematics education and the psychology of mathematics education within teacher preparation, b) a sharing time by both presenters and participants of current in-progress research projects, c) establishing an active network of researchers, and d) a commitment to collaborating and disseminating future research. Through a concerted effort, we will develop a meaningful research agenda that both supports the goals of PME-NA and provides for a strengthened research foundation on reform issues in mathematics education.

## References

Fisher, P. O. & Leitzel, J. R. C. (Eds.). (1996). *Making the change: Pioneering attempts in implementing reform in mathematics teacher preparation.*. Lincoln, NE: University of Nebraska-Lincoln.

Schmidt, W. H. (1996). Release of TIMSS Achievement Results Grades 7/8. [Online] Available http://ustimss.msu.edu/pressre.htm, November 21, 1996.

# PRACTICING TEACHERS BECOMING TEACHER EDUCATORS: WHAT ISSUES ARISE AS TEACHERS CONDUCT SEMINARS FOR THEIR PEERS?

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This session will be centered around a set of issues presented by teachers who have been conducting staff development seminars for their colleagues as part of a four-year Teacher Enhancement Project funded by NSF. The teacher-leaders were all working from the same curriculum for teacher learning which uses teacher-written and video cases to track the development of particular mathematical themes from grades K–6. The curriculum also includes activities that promote participants' explorations of mathematics for themselves, assignments to analyze the mathematical thinking of their own students, opportunities to examine lessons from innovative curriculum, and essays reporting research on related topics.

As teachers became facilitators of this demanding program, they encountered issues which caused them to analyze their responsibilities as teacher educators in conducting staff development. At this session, eight teacher-leaders and two project staff members will present the following issues:

- How do teacher-leaders maintain their goals for the seminar while still responding to the expectations and emotional responses of the participants?
- How can teacher-leaders ensure that negative voices don't take over discussions, while still guaranteeing that genuine confusions and concerns get aired?
- What are the risks/rewards involved as teacher-leaders respond in writing to their peers' work?
- What does it mean to be a primary-grade teacher taking responsibility for challenging the mathematical ideas of upper-grade colleagues?

The session will be interactive and will include opportunities for small group discussion. We will conclude by addressing the following general questions:

What ideas about the process of teacher change do teacher-leaders develop?

What kinds of support do teachers need to take on the role of a teacher leader?

# PRINCIPLES FOR MIDDLE GRADES TEACHER ENHANCEMENT: A PARTICIPATORY SESSION FOR TEACHER EDUCATORS

**Presenters:**

Kathleen Cramer University of Wisconsin - River Falls - Middle Grades TE - Model Program Development. (NSF)

William Bush University of Kentucky - Discussant

Richard Lesh University of Massachusetts - Dartmouth - Middle Grades TE - Model Program Development. (NSF)

Thomas R. Post University of Minnesota - Organizer - Middle Grades TE - Model Program Development. (NSF)

Sid Rachlin University of North Carolina - Greenville - Middle Math Project (NSF)

Barbara Reys University of Missouri - Columbia - Middle Grades TE Project (NSF)

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As new elementary and middle school NSF curricula become available (1995 -1997) and additional demands are made upon classroom teachers to teach different mathematics differently, members of the mathematics education community will be called upon to provide appropriate staff development activities designed to enable teachers to effectively meet these new and complex demands. The need for continuing professional development for teachers will be especially apparent in these new curricula as different assumptions are made about the nature of mathematics, problem solving and learning.

Are there (should there be?) guiding principles to help in the conceptualization and implementation of these new teacher enhancement efforts? What has been learned during the course of our ongoing work with middle grades teachers that would prove useful to other teacher educators with similar goals? How does teachers' specific mathematical knowledge relate to their performance in the mathematics classroom and to the mathematical integrity of the lessons conducted with students?

Each of the participants has been involved in one or more NSF teacher enhancement grants. There will be a good deal of practical wisdom represented.

Cramer, Lesh and Post are concerned with the design of an exportable teacher enhancement program which focuses on the importance of teacher's integrated mathematical, psychological and pedagogical knowledge. It is also designing and testing an evaluation program. Samples of newly developed teacher enhancement and assessment materials will be provided and discussed.

Reys and Reys began a three year teacher enhancement effort in the summer of 1995. Teachers in their project began to use new NSF curricula in the fall of '95. What is the nature of the most successful experiences provided? What are the pitfalls to be aware of? They will share experiences and insights gained from the first and second years of their project.

Rachlin has had a three year grant (Middle Math Project) to work with teacher educators responsible for providing mathematics courses to prospective and in-service middle grades teachers. Each of the participants was required to 're-do' and teach one such course. His comments will focus on the problems and successes encountered with this effort and will provide an important contrast to other session foci which will center on work conducted directly with classroom teachers.

Bush has been involved in a variety of NSF projects related to teacher enhancement and is a frequent panel reviewer in Washington DC. He will attempt to place this discussion in the larger context of general middle grades teacher enhancement needs and concerns.

# UNDERGRADUATE MATHEMATICS FOR ELEMENTARY TEACHERS

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One of the main challenges in implementing mathematics education reform is teacher preparation. The elementary preservice teachers are a particular challenge because of their weak knowledge base, beliefs about mathematics and mathematics teaching, and often anxious attitudes toward mathematics.

Just as theoretical frameworks in constructivism have influenced how we study student learning in mathematics, they have changed the directions in research on teaching.

This growing body of research now includes Shulman's (1986, 1987) general framework of teacher knowledge and Ball's (1990) knowledge structure of mathematics teachers in particular. There has been a mass of studies ranging from Leinhardt and Smith (1985) to Zakis and Campbell (1996). In addition to the studies on teacher knowledge, there also has been a focus on teachers' personal beliefs and attitudes about mathematics (Thompson, 1986, 1992; Lubinski, 1994).

While educators continue to study teacher knowledge, attitudes, and beliefs, we are applying what we have learned to teacher education programs. However, what we advocate based on our theory, our programs, or even the reform documents such as the NCTM books is in a precarious position in the swing of the pendulum that represents mathematics education history. As Ball (1996, p. 504) describes, the theory and documents of reform "[are] long on promise and images. However, considerable work lies ahead if the reform ideas are to permeate daily practice in the schools." That work includes studies that demonstrate the efficacy of our theory and programs so that the pendulum will not take another drastic swing.

This discussion session will contribute to the coordination of this work. The leaders of the session are involved with statewide reform efforts. They will present the working details and studies of elementary teacher preparation programs with a particular focus on the mathematics content courses. They are inviting particular participants representing regions ranging from the Southwest to the Northeast to add to this discussion with research findings and program results.

This panel represents a cross-section of the mass of ongoing research on teacher knowledge and teacher preparation. It is hoped that the invited participants will begin a discussion that other participants in the session can use as a springboard for presentations of their current related work. The result of this discussion group will be an exchange of ideas and research information that will establish contacts to promote and stimulate collaborative research in teacher preparation and development. The teacher education community needs to establish organized reporting of the efficacy of its theories and programs.

## References

Ball, D. L. (1996) Teacher learning and the mathematics reforms: What we think we know and what we need to learn. *Phi Delta Kappan, 3*, 500-508.

Ball, D. L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for research in mathematics education, 21*(2), 132-144.

Cooney, T. J. (1988). The issue of reform: What have we learned from yesteryear? *The Mathematics Teacher, 81*(5), 352-363.

Cooney, T. J. (1980). Research on teaching and teacher education. In R. Shumway (Ed.), *Research in mathematics education*. (pp. 433-474). Reston, VA: NCTM.

*Guidelines for the mathematical preparation of prospective elementary teachers*. (1996). Austin, TX: The Texas Statewide Systemic Initiative.

Lampert, M. (1988). Reinventing the meaning of "knowing" in mathematics. *Communication quarterly, 2*(1).

Leinhardt, G. (1989). Math lessons: A contrast of novice and expert competence. *Journal of Research in Mathematics Education, 20*(1), 52-75.

Leinhardt, G., & Smith, D. A. (1985). Expertise in mathematics instruction: Subject matter knowledge. *Journal of Educational Psychology, 77*, 247-271.

Lubinski, C., and Vacc, N. (1994). Implementing the Professional Standards for Teaching Mathematics: The influence of teachers' beliefs and knowledge on learning environments. *Arithmetic Teacher, 41*(8), 476-479.

McGehee, J. J. (1990). *Prospective secondary teachers' knowledge of the function concept*. Unpublished doctoral dissertation, University of Texas, Austin.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: The Council.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 100-121.

Thompson, A. G. (1992). Teachers' beliefs and conceptions: A synthesis of the research. In D. Grouws (Ed.). *Handbook of research on mathematics* *teaching and learning* (pp. 127-146). New York: Macmillan Publishing Company.

Thompson, A. G. (1984). The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice. *Educational Studies in Mathematics, 15*, 105-127.

Zazkis, R., & Campbell, S. (1996) Divisibility and multiplicative structure of natural numbers: Preservice teachers' understanding. *Journal of Research in Mathematics Education, 27*(5), 540-563.

# TEACHER KNOWLEDGE

# A MODEL FOR STUDYING THE RELATIONSHIP BETWEEN TEACHERS' COGNITIONS AND THEIR INSTRUCTIONAL PRACTICE IN MATHEMATICS

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The purpose of this exploratory study was to develop a model that uses a cognitive perspective within a problem-solving framework to examine the instructional practice and underlying cognitions of teachers of secondary school mathematics. To examine instructional practice, a Phase-Dimension Framework for the Assessment of Mathematics Teaching was developed. To examine the related thoughts of teachers before, during and after lesson enactments a Teacher Cognitions Framework was developed. The value of the model resides in the interrelationship of the two frameworks which can yield information about the role of teacher cognition in instructional practice. The model shows promise as an assessment tool for researchers, teacher educators, and supervisors who wish to better understand and thereby influence the quality of instructional practice in mathematics.

## Objectives

The purpose of this exploratory study was to design a model for assessing teachers' instructional practices in mathematics and the cognitions associated with these practices. The model we developed comprises two frameworks. One framework allows for the systematic assessment of instructional practice in mathematics using dimensions of lessons (tasks, learning environment, discourse) as articulated in the Professional Standards for Teaching Mathematics (NCTM, 1991) and supported by research on the structural components of teaching (Jones, Palincsar, Ogle, & Carr, 1987). Using this framework we examined the three phases of each lesson (initiation, development, closure). The other framework allows for the assessment of the full range of teacher cognitions including teacher knowledge, beliefs, and goals across three stages of teaching: preactive (planning), interactive (monitoring and regulating), and postactive (evaluating and revising). The model formed a basis for the systematic study of the relationship between teacher cognitions and their instructional practice and showed promise as a framework for evaluating mathematics teaching within the current reform movement.

## Theoretical Framework

Within the last two decades, researchers have broadened their lens of inquiry by moving beyond the mere examination of teacher behaviors to studying teacher cognitions (Ernest, 1988; Shavelson, 1986; Shulman, 1986). Researchers have found consistent findings on the differences in the cognitions and instructional practices of expert and novice teachers (Borko & Livingston, 1989; Leinhardt, 1989; Livingston & Borko, 1990). Using a conception of teaching as problem solving, researchers have shed further light on the relationships between cognitions and instructional practice in mathematics (Carpenter, 1989; Fennema, Carpenter & Peterson, 1989). Researchers have identified some of the critical components of teacher cognitions as teacher knowledge, beliefs, goals, and thought processes (Ball, 1991; Cobb, Yackel & Wood, 1991; Fogarty, Wang & Creek, 1983; Peterson, Fennema, Carpenter & Loef, 1989). Although previous investigations have called attention to the importance of cognitions and behavior in the study of teachers, more needs to be done to examine teacher cognitions as an integrated whole. Furthermore, there has been an absence of priori criteria against which to judge the quality of instructional practice. Kagan (1990) and Leinhardt (1990) have identified this issue as the "ecological validity" or "performance verification" problem in research on teaching from a cognitive perspective. Koehler and Grouws (1992), in their recent review of the literature on mathematics teaching practices and their effects, observed that instructional quality was a topic that researchers have avoided, and recommended that it should be more adequately addressed in research on mathematics teaching. We have tried to be responsive to these previously unaddressed issues by systematically assessing teacher cognitions and their instructional practice in mathematics.

## Methods

The authors and a research assistant observed and videotaped fourteen secondary school teachers teaching a mathematics lesson of their own design. At the conclusion of their lesson each teacher participated in: a) a postlesson structured interview, followed by (b) a stimulated-recall interview as they viewed the videotape of their lesson, followed by (c) a debriefing interview. Audiotapes were made of all interviews. Audiotapes and the audio part of the videotapes were transcribed for analysis.

The instructional practice was assessed based on the application of the Phase-Dimension Framework for the Assessment of Instructional Practice in Mathematics (PDF). The framework consisted of 9 elements describing the three major dimensions of instruction: Tasks (modes of representation, motivational strategies, sequencing/difficulty level); Learning Environment (social/intellectual climate, modes of instruction/pacing, administrative routines); and Discourse (teacher-student interaction, student-student interaction, questioning) Each lesson received three dimension scores for each of the three phases of the lesson. The average of the sum of the nine scores and the respective standard deviations resulted in the categorization of the lessons as either Student Centered, Teacher Centered, or Mixed.

**Table 1** *Lesson Dimensions and Dimension Indicators*

| Dimensions | Description of Dimension Indicators |
|---|---|
| **Tasks** | |
| Modes of Representation | Provides such representations as symbols, diagrams, manipulatives, computer or calculator representations accurately to facilitate content clarity. Provides multiple representations that enable students to connect their prior knowledge and skills to the new mathematical situation. |
| Motivational Strategies | Provides tasks that capture students' curiosity and inspires them to speculate and to pursue their conjectures. The diversity of student interests and experiences must be taken into account. The substance of the motivation is aligned with the goals and purposes of instruction |
| Sequencing/Difficulty Level | Sequences tasks such that students can progress in their cumulative understanding of a particular content area and can make connections among ideas learned in the past to those they will learn in the future. Uses tasks that are suitable to what the students already know and can do and what they need to learn or improve on. |
| **Learning Environment** | |
| Social/Intellectual Climate | Establishes and maintains a positive rapport with and among students by showing respect for and valuing students' ideas and ways of thinking. Enforces classroom rules and procedures to ensure appropriate classroom behavior. |
| Modes of Instruction/Pacing | Uses instructional strategies that encourage and support student involvement as well as facilitate goal attainment. Provides and structures the time necessary for students to express themselves and explore mathematical ideas |

| | |
|---|---|
| | and problems.AdministrativeRoutinesUses effective procedures for organization and management of the classroom so that time is maximized for students' active involvement in the discourse and tasks. |
| **Discourse** | |
| Teacher-Student Interaction | Communicates with students in a non-judgmental manner and encourages the participation of each student. Requires students to give full explanations and justifications or demonstrations orally and/or in writing. Listens carefully to students' ideas and makes appropriate decisions regarding when to offer information, when to provide clarification, when to model, when to lead and when to let students grapple with difficulties. |
| Student - Student Interaction | Encourages students to listen to, respond to, and question each other so that they can evaluate and, if necessary, discard or revise ideas and take full responsibility for arriving at mathematical conjectures and/or conclusions. |
| Questioning | Poses variety of levels and types of questions using appropriate wait times that elicit, engage and challenge students' thinking. |

The Teacher Cognitions Framework (TCF) was used to assess teachers' thoughts through an analysis of the interviews and the lesson plans. For each teacher: (a) preactive cognitions (lesson planning) were assessed from the lesson plan and the transcription of the postlesson interview; (b) interactive cognitions (monitoring and regulating) were assessed from the transcription of the stimulated-recall interview; and (c) postactive cognitions (evaluating and revising) were assessed from the transcriptions of the debriefing interview. The overarching cognitions (knowledge, beliefs, goals) were assessed through the lesson plans and the transcriptions of all three interviews.

**Table 2** *Components of Teacher Cognitions and Description of Indicators*

| Cognitions | Description of Indicators of Cognition |
|---|---|
| **OVERARCHING** | |
| **Goals** | To help students construct their own meaning so that they will develop conceptual, as well as procedural understanding and will value the mathematics and feel confident in their abilities. |
| **Cognitions** | **Description of Indicators of Cognition** |
| **Knowledge** | |
| Pupils | Has specific knowledge of pupils' prior knowledge and experiences, abilities, attitudes and interests. |
| Content | Has conceptual and procedural understandings of the content and is aware of and appreciates the connections among it and past and future areas of study. |
| Pedagogy | Has understanding of how students learn mathematics that guides them in developing suitable teaching strategies and anticipating and preparing for areas of difficulty. |
| **Beliefs** | |
| Pupils | Views the role of students as active participants in their own learning. They should make conjectures, propose approaches and solutions to problems, debate the validity of one another's claims, and verify, revise and discard ideas on the basis of their own and other students' mathematical reasoning. |
| Content | Views mathematics as a "dynamic and expanding system of connected principles and ideas constructed through exploration and investigation." (NCTM, 1991, p. 133) |
| Pedagogy | Views teacher's role as one of a facilitator of student learning through selections of prob- |

| | |
|---|---|
| | lem-solving tasks and the leading and orchestration of communication in which students are challenged to think for themselves through mathematical reasoning. |
| **PREACTIVE** | |
| **Planning** | The focus of the lesson is on building conceptual understanding based on what the students already know, and focusing on mathematical processes underlying the procedures to be developed as well as the skill development required by the content specifications.<br>Tasks are logically sequenced to build on previous student understanding and are appropriate for clarifying new concepts and arousing students' interest and curiosity. |
| **INTERACTIVE** | |
| **Monitoring** | Observes, listens to, and elicits participation of students on an ongoing basis in order to assess student learning and disposition toward mathematics. |
| **Regulating** | Adapts or changes plans while teaching based on the information received through monitoring student learning and interest. |
| **POSTACTIVE** | |
| **Evaluating** | Describes and comments on students' understanding of concepts and procedures and dispositions toward mathematics as well as the effects of their instruction on these outcomes. |
| **Revising** | Uses information from their evaluation of student learning and instructional practices to revise and adapt their subsequent plans for instruction. |

For each category of lesson quality determined by the PDF, we examined the data from the TCF to see if any patterns emerged. We then described the patterns of cognitions associated with each category of lesson.

## Data Sources

The subjects for this study included seven beginning and seven experienced teachers of secondary school mathematics. All but one of the teachers taught in urban schools. All taught at grade levels ranging from 9 to 11 and were observed teaching one lesson. Data were obtained from observation narratives, videotapes of the lessons, audiotapes of interviews and lesson plans of the teachers.

## Results

Through the use of the model we were able to assess the instructional practice and the related cognitions of fourteen teachers of secondary school mathematics. Using the PDF we were able to assess and evaluate the lessons. Five lessons were categorized as Student Centered (four taught by experienced teachers, one taught by a beginner); four of the lessons were categorized as Teacher Centered (all four taught by beginners); and five lessons were categorized as Mixed (three taught by experienced teachers, two taught by beginners). Student Centered lessons received high ratings in all nine phase-dimension scores. Mixed lessons received inconsistent ratings on the nine scores and Teacher Centered lessons received low ratings in all nine scores.

Through the use of the TCF we were able to describe the 14 teachers' cognitions during three stages of instruction: preactive, interactive and postactive. Patterns of cognitions were found when the TCF results were organized according to the PDF ratings.

For the Student Centered lessons the instructional practice was characterized by a consistent interplay of tasks, learning environments and discourse which contributed towards students' active involvement in mathematical explorations throughout all phases of the lessons. A similar consistency of results was noticed in the cognitions revealed by these teachers. That is, their knowledge, beliefs and goals all centered around student learning with understanding, as did their thought processes before, during and after the lesson.

For the Teacher Centered lessons the instructional practice was characterized by a consistent interplay of tasks and learning environments which contributed towards students' passive role in the classroom where minimal discourse between teacher and students and between students and students was evident. The descriptive analysis revealed that the teacher expressed cognitions that primarily focused on their own practices rather than on student learning. Their knowledge, beliefs and goals centered around content coverage, skill development and classroom management, as were their thought processes before, during and after the lesson.

For the Mixed lessons there was a large range of scores, indicating an inconsistent pattern of lesson quality. A similar inconsistency in the focus of the expressed cognitions of these teachers was revealed in the descriptive analysis. Two teachers introduced tasks that were either too difficult or confusing during the developmental phase of their lessons. At this point the discourse turned from one that had been centered around student input to one that could be characterized as teacher telling alone. Each teacher admitted inadequate knowledge of the content or student readiness for the content. For the other three teachers, unlike the well designed tasks and positive learning environment, the discourse was fast paced and teacher dominated throughout all phases of the lesson. Each of these teachers revealed beliefs that teacher-dominated discourse fostered more efficient content coverage and student understanding.

## Conclusions and Implications for Future Study

Through the use of our model, we were able to assess the teaching of mathematics as an integrated whole and obtain a better understanding of instructional practice and associated teacher cognitions. With further refinement the Phase-Dimension Framework for the Assessment of Instructional Practice and the Teacher Cognitions Framework may prove useful to researchers and teacher educators in their preservice and inservice mathematics programs and to supervisors in their assessment of teachers and their instructional practice. We believe that the model can facilitate better understanding of the psychological aspects of teaching mathematics and it should lead to implications for teacher education.

This research was partly supported by a PSC-CUNY Research award under grant No. 668416.

## References

Ball, D.L. (1991). Research on teaching mathematics: Making subject matter knowledge part of the equation. In J.E.Brophy (Ed.), *Advances in research on teaching: Teachers' subject matter knowledge and classroom instruction* (pp. 1-48) (Vol. 2). Greenwich, CT: JAI Press.

Borko, H. & Livingston, C. (1989). Cognition and improvisation: Differences in mathematics instruction by expert and novice teachers. *American Educational Research Journal, 26*(4), 473-498.

Carpenter, T.P. (1989). Teaching as problem solving. In R. Charles & Silver (Eds.), *The teaching and assessing of mathematical problem solving* (pp. 187-202). Reston,VA: NCTM.

Cobb, P., Yackel, E., & Wood, T. (1991). Curriculum and teacher development: Psychological and anthropological perspectives. In E. Fennema, T. Carpenter, & S. J. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 55-82). Albany, NY: State University of New York Press.

Ernest, P. (1988, July). The impact of beliefs on the teaching of mathematics. Paper prepared for ICME VI, Budapest, Hungary.

Fennema, E., Carpenter, T.P., & Peterson, P. L. (1989). Teachers' decision making and cognitively guided instruction: A new paradigm for curriculum development. In N.F. Ellerton & M. A. (Ken) Clements (Eds.), *School mathematics: The challenge to change* (pp. 174-187). Geelong, Victoria, Australia: Deakin University Press.

Fogarty, J., Wang, M., & Creek, R. (1983). A descriptive study of experienced and novice teachers' interactive instructional thoughts and actions. *Journal of Educational Research, 77*, 22-32.

Jones, B. F., Palincsar, A. S., Ogle, D. S., Carr, E. G. (1987). Strategic teaching: A cognitive focus. In B. F. Jones, A. S. Palincsar, D. S. Ogle, & E. G. Carr (Eds.), *Strategic teaching and learning: Cognitive instruction in the content areas* (pp.33-63). Alexandria, VA: ASCD.

Kagan, D. M. (1990). Ways of evaluating teacher cognition: Inferences concerning the Goldilocks principle. *Review of Educational Research, 60*(3), 419-469.

Koehler, M. S. & Grouws, D. A. (1992). Mathematics teaching practices and their effects. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 115-146). New York: Macmillan Publishing Company.

Leinhardt, G. (1990). Capturing craft knowledge in teaching. *Educational Researcher, 19*, 18-25.

Leinhardt, G. (1989). Math lessons: A contrast of novice and expert competence. *Journal for Research in Mathematics Education, 20*(1), 52-75.

Livingston, C. & Borko, H. (1990). High school mathematics review lessons: Expert-novice distinction. *Journal for Research in Mathematics Education, 21*, 372-387.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics.* Reston, VA: The Council.

Peterson, P. L., Fennema, E., Carpenter, T. P., & Loef, M. (1989). Teachers' pedagogical content beliefs in mathematics. *Cognition and Instruction, 6*, 1-40.

Shavelson, R.J. & Stern, P. (1981). Research on teachers' pedagogical thoughts, judgments, decisions and beliefs. *Review of Educational Research, 51*,455-498.

Shulman, L. S. (1986a). Paradigms and research programs in the study of teaching: A contemporary perspective. In M.C. Wittrock (Ed.), *Handbook of research on teaching* (3rd ed. , pp.3-36). New York, NY: Macmillan.

# MEDIATING PEDAGOGICAL CONTENT KNOWLEDGE THROUGH SOCIAL INTERACTIONS: A PROSPECTIVE TEACHER'S EMERGING PRACTICE

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This is a preliminary investigation of one of three case studies conducted in prospective middle school teachers' mathematics classrooms. A Vygotskian perspective was adopted to understand the prospective teacher's construction of pedagogical content knowledge during the student teaching practicum. In particular, we considered how her knowledge about teaching mathematics was mediated through interactions with various social agents. Our attempts to help the prospective teacher bridge her zone of proximal development suggested that guidance by a more knowing other has a central influence on the prospective teacher's emerging practice.

## Introduction

In recent years, the preeminence of constructivism as an epistemological orientation in mathematics education has directed research toward understanding students' constructions of mathematical knowledge (e.g., Cobb, Yackel, & Wood, 1992; Steffe & Tzur, 1994). Pursuant to this are ongoing national reform efforts for teaching K-12 mathematics that reflect our understanding of students' thinking about mathematics. According to Simon (in press), the success of such reforms hinges on the preparation of a professional cadre of mathematics teachers in accordance with a strong reform-minded research base on teacher development. However, our understanding of how prospective teachers construct their knowledge about teaching mathematics in situ is currently underdeveloped. Cobb, Yackel, and Wood (1991), recognizing the importance of the classroom as a learning environment for teachers, speculate that "teachers should be helped to develop their pedagogical content knowledge and beliefs in the context of the classroom practice" (p. 90). Until the student teaching practicum, prospective teachers' understanding of how to teach mathematics is almost necessarily academic. They are primarily confined to university settings which may offer only decontextualized opportunities for developing their craft. The professional semester suggests the optimal context in which knowledge of mathematics and mathematics teaching and learning coalesce into an emerging practice for the neophyte teacher.

## A Vygotskian Perspective for Teacher Development

To understand how prospective teachers construct their pedagogical content knowledge, we appeal to the theoretical lens of social constructivism as articulated by Vygotsky. Vygotsky's belief in the social origins of higher mental functioning embeds human consciousness "in the external processes of social life, in the social and historical forms of human existence" (Luria, 1981, as cited in Wertsch & Tulviste, 1996, p. 54). This belief, coupled with his emphasis on semiotic mediation, underlies Vygotsky's argument that social interactions are the basis for an individual's development. According to Minick (1996), Vygotsky maintained that "higher voluntary forms of human behavior have their roots in social interaction, in the individual's participation in social behaviors that are mediated by speech" (p. 33). Indeed, Vygotsky's (1986) assertion that higher mental functions are directly mediated through social interactions invites specific contexts for investigating the prospective teacher's transition from mathematics student to mathematics teacher during the professional semester. Such contexts include the mathematics classroom assigned to the prospective teacher, meetings between the prospective teacher and professional support personnel, as well as opportunities for reflection by the prospective teacher. Viewing mind metaphorically as social and conversational, Ernest (1994) posits that people are "formed through their interactions with each other (as well as by their internal processes) in social contexts" (p. 69). This is no less true for prospective teachers during the student teaching practicum. Consequently, we have considered how one prospective teacher's pedagogical content knowledge is mediated during the professional semester through meaningful social interactions with her students, university supervisor, and self.

For our investigation, Vygotsky's (1978) construct of the zone of proximal development lends theoretical support to guiding the prospective teacher as her practice emerges. In particular, it upholds the use of intentional instruction during the supervisory process to effect further mediation of the prospective teacher's pedagogical content knowledge. According to Manning and Payne (1993), "the mechanism for growth in the zone is the actual verbal interaction with a more experienced member of society. In the teacher education context, this more experienced person is likely to be a supervising teacher, college supervisor, teacher educator, or a peer who is at a more advanced level in the teacher education program" (as cited in Jones, Rua, & Carter, 1997, p. 6). Jones and colleagues suggest that such assistance by a more experienced person can be given through "prompting, modeling, explaining, asking leading questions, discussing ideas, [and] providing encouragement" (p. 4). One of Vygotsky's central propositions, the zone of proximal development is unique in that it "connects a general psychological perspective on...development with a pedagogical perspective on instruction" (Hedegaard, 1996, p. 171). As such, it has powerful implications for changing higher mental functioning on both the interpsychological and intrapsychological planes (Wertsch & Tulviste, 1996).

## The Nature of Inquiry

Our socio-cultural perspective led us to the dynamics of Mary Ann's seventh grade mathematics classroom for a case study of this prospective teacher. From our first meeting in which we invited her participation in this study and explained the obligations of one of the researchers as her university supervisor, Mary Ann's enthusiasm promised a partnership from which we all could learn. While her academic journey in a four-year teacher preparation program chronicled a multitude of diverse experiences for one who had chosen a dual concentration in mathematics and science, her anticipation of student teaching signaled a readiness to move beyond the safety of academe.

Mary Ann agreed to let us visit her classroom weekly during the student teaching practicum. We conceptualized the nature of these visits as an extension of Steffe's (1991) constructivist teaching experiment. That is, rather than eliciting models of children's constructions of mathematical knowledge, we used the teaching experiment to investigate a prospective teacher's construction of pedagogical content knowledge. Each visit was an audio- and videotaped three-hour sequence which began with observing Mary Ann teach her first period general mathematics class. The observation focused on episodes of discourse in the classroom that prompted her to rethink how she taught mathematics or indicated to us her need to do so. Since Mary Ann's planning period was the following hour, we then collaborated in a teaching episode to make sense of the events of the previous class and plan alternative strategies for subsequent lessons. Immediately after the teaching episode, we observed Mary Ann teaching the same subject to her third period class in order to document effects of the teaching episode. Finally, Mary Ann was asked to write personal reflections after each visit describing how her teaching practice had changed through this process.

## Developing Pedagogical Content Knowledge through Classroom Practice

Our first visits served to build Mary Ann's trust that we were collaborators in developing her understanding about teaching mathematics. For us, this was pivotal if we were to help bridge her zone of proximal development. Soon, the amicable rapport she enjoyed with her students spilled into our relationship. Mary Ann's interactions with her students during these early observations revealed a focus for our teaching episodes which one particular visit brought to light. During her first period class, she introduced a lesson on "working backwards" as a problem solving technique by giving students a problem to solve on their own. "I'm thinking of a number", she said, "that if you divide by three and then add five, the result is eleven." The discourse that followed likely typifies the novice teacher's practice. After a short pause, Mary Ann began to dole out hints until a correct answer appeared. A student was asked to share her procedure for obtaining this solution, upon which Mary Ann began a step-by-step account of how to work backwards to find the answer. Her instructions were interspersed with questions which required students only to perform simple computations, or that were worded so as to suggest the desired response. She later commented to us, "I look at math as just operations you go through, just like a series of steps." Mary Ann seemed to interpret students' participation as an indication of understanding; however, her frustration surfaced when the class attempted to solve an almost identical problem.

Mary Ann: OK, I'm thinking of a number if you divide by three and then add five, the result is thirteen. So what would I first do just to get an idea of what we're talking about? Does anybody know how we did the last one? (No one responds to her questions.) OK, what we need to do first, step one, we need to write everything down in the order in which we read it. So we start reading, 'If you divide by three', so we divide by three. Then we're gonna [sic] add five. . . . Then the result is thirteen, and we want to work backwards. So what have we got to do when we work backwards? (Again, no response.) OK, what was the word that we used when we talked about what we've got to do with all of these [operations]?

Class: Inverse.

Mary Ann: Inverse. OK, so we have to take the inverse of operations . . . OK? So what are we going to do with the five? Add it or subtract it?

Class: Subtract.

Mary Ann: Subtract. We're working with inverse. We're working with opposites. OK, then are we gonna [sic] divide or multiply by three?

Class: Multiply.

Mary Ann: Multiply by three. OK, that's step two, to write down everything that's the inverse, and it's very important that you keep the same order. You have to keep the same order as the problem. So it will help if you write it down, like this. Step one, step two...(She points out each of the steps that she has written on the overhead projector.) OK, step three is to actually solve the problem. What are we gonna [sic] do? Somebody tell me what the first step...[is].

Mary Ann continued this interaction with one particular target student. After he produced a response of twenty-four, she concluded, "Twenty-four. So that's my answer. That is the answer. I ask you what number did I start with, you'll say what?" The students are silent. She continues, "What number did I start with? The problem says, 'I'm thinking of a number'. What number am I thinking of?" Hesitantly, they began to suggest various numbers that occurred in the problem. Twenty-four seemed to dominate, cueing Mary Ann to once again argue its veracity. She repeated, "Twenty-four. That is your answer. You worked backwards. You said thirteen minus five is eight and eight times three is twenty-four. That is the number you started with." It seemed to us that students were not actively solving a problem, but were at most reproducing a series of steps. Also, it was not clear that they felt any sense of ownership in this process. Mary Ann insisted that they had worked the problem while, in truth, one student had volunteered most of the responses to her running dialogue. Our challenge was to use these classroom interactions to help her develop a sense of mathematics as a problem solving endeavor, to let students struggle with unfamiliar problems, to promote their justifications of ideas through mathematical discourse with each other and Mary Ann, and to learn from unsuccessful attempts at problem solving. The students' mathematical dilemmas had already prompted Mary Ann to reconsider how she taught the lesson, presenting us with an opportunity to guide her emerging practice. During the teaching episode, we planned an approach with Mary Ann that would prioritize students' obligations as problem solvers.

The success of the next class rested on Mary Ann's willingness to risk a different approach on a topic with which she was admittedly uncomfortable. What happened was as powerful for us as for her. Departing from her previous strategy, Mary Ann placed the students in dyads to solve the problem she had started with in her earlier class. She removed herself as the sole authority of knowledge, delaying closure so that students would begin to communicate mathematically with each other. Sharing ideas was no longer limited to those who had found the right answer. As one of the students began explaining her group's strategy, Mary Ann looked to us in excited disbelief and mouthed, "Wow!" She praised the student, "You just taught our lesson for today!" Her journal reflection for this visit succinctly illustrated the transformation in her knowledge about teaching mathematics:

> Teaching this [to the first class] was a real eye-opener for me. I think I totally confused my students completely. I tried to show them steps without letting them think about the problem themselves. . . .[The next class] was different. After [the university supervisor] and I talked about the lesson and going over several suggestions, things seem [sic] to run much smoother. Instead of throwing information out, I let them figure the problem out in their own style. . . . To my surprise, one of my students performed the problem exactly as the strategy suggested. Boy,

> was this a memorable event. The pressure was lifted off of me. . . Once the students saw how one of their peers was able to solve the problem, things were a lot more clear to all. I learned that having a student come up with the solution means more to the others than the teacher giving a long, drawn out lecture. Sometimes you need for things to flop, so you can think up new ways to approach the situation.

## Conclusions

Although Mary Ann's emerging pedagogical content knowledge exhibited a certain nonlinearity over the semester, there were indications that our visits were moving her practice in a positive direction. The visit described here, later monikered the "problem solving day", served to anchor her ability to try alternative strategies. She seemed to receive from us the permission to take risks regardless of the outcome. We felt that this was an interesting parallel to her own practice. Although our three-hour schedule with Mary Ann was ideal for us, we realize that it may have limitations for supervising large numbers of prospective teachers. However, helping Mary Ann to make sense of classroom events as they unfolded seemed crucial to her development. As we encouraged her to take meaningful risks in her teaching, Mary Ann was able to move beyond what we believe she would have accomplished on her own.

## References

Cobb, P., Yackel, E., & Wood, T. (1991). Curriculum and teacher development: Psychological and anthropological perspectives. In E. Fennema, T. P. Carpenter, & S. J. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 83-119). Albany, NY: State University of New York.

Cobb, P., Yackel, E., & Wood, T. (1992). Interaction and learning in mathematics classroom situations. *Educational Studies in Mathematics, 23*(1), 99-122.

Ernest, P. (1994). Social constructivism and the psychology of mathematics education. In P. Ernest (Ed.), *Constructing mathematical knowledge: Epistemology and mathematical education* (pp. 62-72). London: Falmer.

Hedegaard, M. (1996). The zone of proximal development as a basis for instruction. In H. Daniels (Ed.), *An introduction to Vygotsky* (pp. 171-195). London: Routledge.

Jones, G., Rua, M., & Carter, G. (1997). *Science teachers' conceptual growth within Vygotsky's zone of proximal development.* Paper presented at the meeting of the American Educational Research Association, Chicago, IL.

Minick, N. (1996). The development of Vygotsky's thought. In H. Daniels (Ed.), *An introduction to Vygotsky* (pp. 28-52). London: Routledge.

Simon, M. (in press). Developing new models of mathematics teaching: An imperative for research on mathematics teacher development. In E. Fennema & B. Nelson, (Eds.), *Mathematics teachers in transition*.

Steffe, L. (1991). The constructivist teaching experiment: Illustrations and implications. In E. Von Glasersfeld (Ed.), *Radical constructivism in mathematics education* (pp. 177-194). Dordrecht, The Netherlands: Kluwer.

Steffe, L., & Tzur, R. (1994). Interaction and children's mathematics. In P. Ernest (Ed.), *Constructing mathematical knowledge: Epistemology and mathematical education* (p. 8-32). London: Falmer.

Vygotsky, L. (1978). *Mind in society*. (M. Cole, S. Scribner, V. John-Steiner, & E. Souberman, Trans.). Cambridge, MA: Harvard University. (Original work published 1934.)

Vygotsky, L. (1986). *Thought and language*. (A. Kozulin, Trans.). Cambridge, MA: Massachusetts Institute of Technology. (Original work published 1934.)

Wertsch, J., & Tulviste, P. (1996). L. S. Vygotsky and contemporary developmental psychology. In H. Daniels (Ed.), *An introduction to Vygotsky* (pp. 53-74). London: Routledge.

# LEARNING TO TEACH ALGEBRAIC DIVISION FOR UNDERSTANDING: A COMPARISON AND CONTRAST BETWEEN TWO EXPERIENCED TEACHERS

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In this paper we examine the explanations that two teachers, Mr. Kantor and Mr. Taylor, constructed about the definition of division in terms of conceptual and procedural knowledge. We also attempt to understand the sources of difficulties that one teacher experienced when teaching this topic. It was found that both teachers' explanations and representations involved procedural and conceptual knowledge but conceptual knowledge dominated Mr. Kantor's explanations and procedural knowledge dominated Mr. Taylor's explanations. Only Mr. Kantor experienced major difficulties due, in part, to the conceptual emphasis and to students' functional fixedness.

In this paper we compare and contrast the explanations and representations that two teachers, Mr. Kantor and Mr. Taylor, constructed for teaching the definition of division. The objectives are to attempt to understand: (a) the role of the teachers' knowledge of representations on their classroom instruction, (b) the difficulties that the teachers face when teaching division for both procedural and conceptual knowledge (Hiebert & Lefevre, 1986). To this end, we address the following research questions: (1) Do Mr. Kantor and Mr. Taylor's explanations involve both procedural and conceptual knowledge? (2) If yes, what representations do these teachers use? (3) Do these teachers experience difficulties when teaching algebraic division? (4) If yes, what are the sources of these difficulties?

## Empirical Background

Previous research has examined teachers' knowledge of representations about division (e.g., Ball, 1990; Simon, 1993). In Simon's (1993) study, only 10 of 33 prospective elementary teachers were able to provide a story problem that could be solved by $\frac{3}{4} \div \frac{1}{4}$. However, we need to examine how teachers use their knowledge in teaching situations (Simon, 1993). Borko et al. (1992) examined one teaching episode in which a middle school student teacher failed to construct a conceptual explanation for the algorithm of division of fractions using word problems and pictures for the example of $\frac{3}{4} \div \frac{1}{2}$. Borko et al.

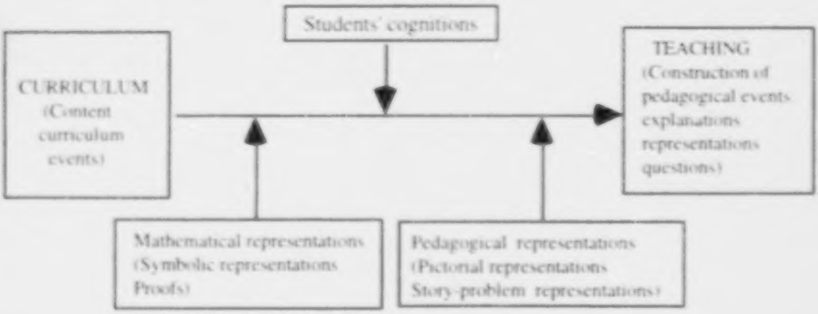
concluded that the student teacher's failure was due, in part, to a lack of knowledge of representations and poor understanding of the algorithm. While these studies suggest that weak knowledge of representations prevents teachers from constructing conceptually based explanations, there is not empirical evidence that knowing the representations of why division is related to multiplication would influence teachers' construction of conceptual explanations. As stated before, this is one of the objectives of this paper. We found that the participants of the study knew representations of why $a \div b = a \cdot \frac{1}{b}, b \neq 0$.

## Conceptual Framework

The figure below displays the conceptual framework that framed the research questions and guided the analysis of the teachers' explanations and representations. A content curriculum event (CCE) is each mathematical idea (e.g., concepts, formulas, theorems, axioms, procedures, etc.) identified in a curriculum text such as curriculum guides or textbooks (Contreras, 1996). Teachers' knowledge of mathematical and pedagogical representations influence their teaching. Students' cognitions may have also an impact on teachers' explanations and representations. [1]

## Methodology and Data Sources

**Participants and setting**. Mr. Kantor and Mr. Taylor teach middle-grade mathematics in a school district known for high student achievement in a U.S. city.


Students' cognitions
CURRICULUM
(Content
curriculum
events)
TEACHING
(Construction of
pedagogical events
explanations
representations
questions)
Mathematical representations
(Symbolic representations
Proofs)
Pedagogical representations
(Pictorial representations
Story-problem representations)

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[1] Because of space limitations, I have only outlined the theoretical framework. A more complete description will be provided during the oral presentation.

At the time of the study, Mr. Kantor and Mr. Taylor had about five and 20 years of experience teaching mathematics, respectively.

**Procedures for data collection**. We videotaped Mr. Kantor and Mr. Taylor's instruction when teaching the lesson of the textbook dealing with the definition of division. Based on a content analysis of the teachers' instruction and the textbook the following CCE's were identified as central to division: the concept of division, the definition of division as $a \div b = a \cdot \frac{1}{b}$, $b \neq 0$, division by zero and the rules of signs for division. Audiotaped interviews were conducted to examine the teachers' knowledge of representations about those CCE's.

## Data Analysis

Content analyses of the transcriptions of the videotapes and interviews was carried out to answer the research questions.

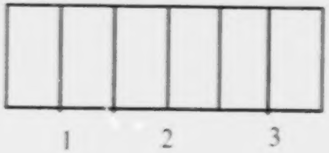
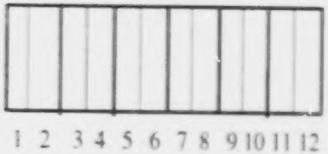
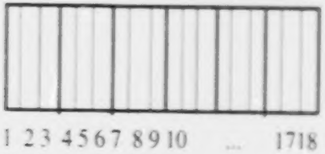
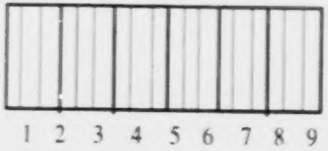
**Teachers' explanations and procedural and conceptual knowledge**. Mr. Kantor constructed explanations for only two CCEs: the concept of division and the definition of division. Mr. Taylor, on the other hand, constructed explanations for three CCE's: concept of division, definition of division, and division by zero. None of the teachers constructed explanations for any of the rules of signs for division. Tables 1 and 2 display the representations that Mr. Taylor and Mr. Kantor constructed for teaching the definition of division, respectively. The tables show that while Mr. Taylor used mainly pictorial representations and the quotitive model for division, Mr. Kantor used mainly story-problem representations and used both the partitive model and the quotitive model for the concept of division.

The data displayed in the tables show that both teachers attempted to teach the definition of division for procedural and conceptual understanding. However, while Mr. Taylor used indirect representations to represent why $a \div b$ as $a \cdot \frac{1}{b}$ using numerical examples and a pattern to establish the plausibility of computing $a \div b$ as $a \cdot \frac{1}{b}$, Mr. Kantor used story problems to illustrate *why* division can be defined in terms of multiplication. For example, Mr. Taylor constructed the following explanations about the representations entered in the definition of division in Table 1:

> Well, another way to think of this … is to think of, well, if I am dividing six, and say how many two's are there? … I want ones, one whole

137

**Table 1** *Mr. Taylor's representations for the concept of division and its relationship to multiplication*

| Concept of division | | Definition of division | | |
|---|---|---|---|---|
| $6 \div 2 = 3$ |  | $\frac{6}{1} \cdot \frac{1}{2}$ | ← Types of parts wholes<br>← How many groups of 2 | $= \frac{6}{2} = 3$ |
| $6 \div \frac{1}{2} = 12$ |   1 2 3 4 5 6 7 8 9 10 11 12 | $\frac{6}{1} \cdot \frac{2}{1}$ | Halves (2 parts)<br>← Groups of 1 | $= \frac{12}{1} = 12$ |
| $6 \div \frac{1}{3} = 18$ |   1 2 3 4 5 6 7 8 9 10 ... 17 18 | $\frac{6}{1} \cdot \frac{3}{1}$ | Thirds (3 parts)<br>← Groups of 1 | $= \frac{18}{1} = 18$ |
| $6 \div \frac{2}{3} = 9$ |  | $\frac{6}{1} \cdot \frac{3}{2}$ | Thirds (3 parts)<br>← Groups of 2 | $= \frac{18}{2} = 9$ |

> units, how many of those pieces do I want? I want two pieces, don't I? So I really gotta say one of two pieces. What about if I want halves? What kinds of parts do I divide this into? I divide it into two parts. If I divide it into two parts then what is the full number of parts I"m gonna end up with? Six whole things, 2 parts per each thing, right? Six times two which is twelve and then I am asking how many of those one halves .... We get groups of ones. How many groups of ones are there in those twelve halves? Well, there are twelve. By doing thirds, I have three parts per each of these wholes, don't I? So I multiply six whole numbers, three parts in each, total, how many parts, 18, and again, how many do I want? I want how many one thirds, so groups of ones again. How many groups of ones are there in 18? There are 18. And the last one is, I divide it into thirds again, so there are how many parts, six wholes times three parts, that's 18, but I am not asking now how many ones parts, I'm asking how many groups of, what, two are in there. So that's another way of thinking of the problem.

We notice that Mr. Taylor is not representing why $a \div b = a(1/b)$ directly. For example, when talking about $6 \div \frac{2}{3}$ he converted the units into thirds to get 18 and since we want groups of two thirds we have $\frac{18}{2}$. From table 1 we see that he was thinking of $\frac{6}{1} \cdot \frac{3}{2} = \frac{18}{2}$ [by multiplication of fractions theorem]. Since both $6 \div \frac{2}{3}$ and $\frac{6}{1} \cdot \frac{3}{2}$ can be expressed as $\frac{18}{2}$ we conclude that $6 \div \frac{2}{3} = \frac{6}{1} \cdot \frac{3}{2}$. However, he did not emphasized that connection. Rather, he used the fact that $6 \div \frac{2}{3}$ and $\frac{6}{1} \cdot \frac{3}{2}$ give the same answer to focus on the pattern between the divisor and the second factor of the multiplication as shown in his explanation:

> What happens is, this [1/2] is just the reciprocal of two, isn't it? This [2] is just the reciprocal of a half. This [3] is just the reciprocal of a third. This [3/2] is just the reciprocal of [two thirds]. So, what we found, by looking at these patterns is that when you want to divide what you can do is change the problem to times the reciprocal of the second number. And, indeed, that's the definition of what division is in algebra. The definition, *a* divided by *b*, I can change it to *a* times the reciprocal of *b*.

After this explanation Mr. Taylor provided several examples to express division in terms of multiplication and he focused on the procedure. Mr. Kantor, on the other hand, focused and emphasized the conceptual base of the connection between $a \div b$ and $a \cdot \frac{1}{b}$ using the *of* model for multiplication (taking a half *of* 7 or $7 \cdot \frac{1}{2}$), the rate model and an indirect approach ($21 \div \frac{3}{4} = \frac{84}{3} = 28$) (see the representations entered under the column "definition of division" in Table 2).

**Teachers' difficulties when teaching the definition of division**. While Mr. Taylor did not experience difficulties when teaching the connection between $a \div b$ and $a \cdot \frac{1}{b}$, Mr. Kantor struggled teaching the connection. To illustrate, Mr. Kantor's struggles to show why $5 \div \frac{1}{2} = 5 \cdot 2$ using the word problem displayed in table 2 are described in the following teaching episode.

1. K: So you have five dollars. You wanna find out how many pencils you can buy that cost fifty cents a piece. … Doesn't it make sense that you get the same answer as five times two? … Using the same situation explain why.

2. S: The opposite.

3. K: Now, in the context of that problem, why does five times two give me the answer?

4. S: Because it equals ten.

5. K: …. six plus four equals ten, too…. but that doesn't mean it's the mechanism to employ this in other problems. What?

6. S: Because when you divide a fraction you can do the same thing by multiplying by the reciprocal.

7. K: Tell me in a concrete way why that gives you the same thing …

8. A: Because it is.

9. K: I can find out how many times it goes into five. The process of doing that, how can I do it?

10. A: Because it is.... Because it is.

11. K: Let me change the problem just a little bit.... I wanna give each kid fifty cents. I have five-dollar bills ... I have to go to the bank and get fifty-cent pieces for five-dollar bills. For every dollar bill they give what?

12. S: Two.

13. K: Two fifty-cent pieces, right? and so that's five times two is how many fifty-cent pieces I get. The samething.... How many times a half goes into five is the same as five times two.

As we can see from this teaching episode, Mr. Kantor is experiencing difficulties getting students to think about why $5 \div \frac{1}{2} = 5 \cdot 2$ using the story problem. Mr. Kantor does not give up and creates another story-problem representation and asked students to explain why $21 \div \frac{3}{4} = 21 \cdot \frac{4}{3}$. However, Mr. Kantor also experienced difficulties this time as evidenced by students' responses and reactions (You can try it, it works; I just told you. You can do it. You like to make things more complicated than they are Mr. Kantor; Oh, boy; You can do it and it works. What else do you want?; because division is multiplication). This time Mr. Kantor represented why $21 \div \frac{3}{4} = \frac{84}{3} = 28$ directly rather than representing it as $21 \div \frac{3}{4} = 21 \cdot \frac{4}{3}$ (See Table 2, second column).

**Explanations and representations for the other content curriculum events.** While Mr. Kantor did not construct explanations nor representations for division by zero, Mr. Taylor constructed a procedural representation: "You can't divide by zero.... That is why ... you can not have ... zero in the denominator." None of the teachers constructed representations about the rules of signs for division.

## Discussion and Conclusion

Although both Mr. Kantor and Mr. Taylor's explanations involved elements of procedural and conceptual knowledge for the definition of division, procedural elements dominated Mr. Taylor's explanations and conceptual elements dominated Mr. Kantor's explanations. In addition, Mr. Taylor's explanation about division by zero was procedural. One possible reason to explain Mr. Taylor's emphasis on procedural knowledge is that he lacks conceptual knowledge about

**Table 2** *Mr. Kantor's representations for the concept of division and its relationship to multiplication*

| Concept of division | | Definition of division | |
|---|---|---|---|
| $7 \div 2$ | S*: You have seven pieces of pizza and you want to divide those among two people. | $7 \div 2 = 7 \cdot \frac{1}{2}$ | S: If you have seven pieces of pizza and each person gets a half *of* that.<br>K: Seven divided by two … is the same thing as taking half *of* it. |
| $5 \div \frac{1}{2}$ | S: How many pencils can you get for five dollars [if each pencil costs] fifty cents? | $5 \div \frac{1}{2} = 5 \cdot 2$ | K: I have to go to the bank and get fifty cent pieces for five-dollars bills. For every dollar bill they give me what? … Two fifty-cent pieces, right? and … five times two is how many fifty-cent pieces I get. |
| $21 \div \frac{3}{4}$ | K: I wanna take some of my students out to lunch and I know I can buy a pop per … seventy five cents a can. I've got twenty one bucks so I have to figure out how many people I can buy pop. | $21 \div \frac{3}{4} = 21 \cdot \frac{4}{3}$ | K: I go to the bank and I say give me twenty one dollars worth in quarters. That's what that is, eighty four quarters and then what's this $\left[\frac{84}{3}\right]$? I put it into groups of three …. I end up with twenty eight groups of three $\left[\frac{84}{3} = 28\right]$, all right? |

S: a student, K: Mr. Kantor

the definition of division and about division by zero. However, interviews revealed that he knew representations about why division can be defined in terms of multiplication. In the case of division by zero, he knew why division by zero is impossible when the numerator is not zero. Therefore lack of conceptual knowledge is not a complete explanation about Mr. Taylor's instructional representations. Another plausible explanation is that Mr. Taylor does not value teaching for conceptual knowledge. However, this explanation might to be rule out because he said that teaching for understanding was very important. Other reasons such as pressure to cover the curriculum, lack of time, etc. may have an influence on his curricular and pedagogical decisions. Those and other reasons may suggest why Mr. Kantor and Mr. Taylor's did not construct explanations about the rules of signs for division.

Mr. Kantor experienced difficulties teaching the definition of division due, in part, to the fact that his explanations involved strong elements of conceptual knowledge. Another factor was students' reluctance to think of the definition of division in conceptual terms. We think that students were reluctant to learn why $a \div b = a \cdot \frac{1}{b}$ because they already knew that connection in procedural terms. This phenomenon is similar to what Gestalt psychologists have termed *functional fixedness*. In Hiebert & Carpenter's (1992) words, "when a particular … procedure is practiced it can become fixed, making it difficult to think of the problem in another way" (p. 79).

## References

Ball, D. L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education*, 21, 132-144.

Borko, H., Eisenhart, M., Brown, C., Underhill, R., Jones, D., & Agard, P. (1992). Learning to teach hard mathematics: Do novice teachers and their instructors give up too easily? *Journal for Research in Mathematics Education, 23*(3), 194-222.

Contreras, J. (1996). The teacher as constructor of pedagogical events about content curriculum events: Toward an empirical validation of a theoretical framework integrating curriculum, teaching, and learning in studies of learning to teach. In E. Jakuboski, D. Watkins, & H. Biske (Eds.), *Proceedings of the Eighteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, 465-470). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Hiebert, J., & Carpenter, T. (1992). Learning and teaching with understanding. In D. A Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390-419). New York: Macmillan.

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1-27). Hillsdale, NJ: Lawrence Erlbaum.

Simon, M. (1993). Prospective elementary teachers' knowledge of division. *Journal for Research in Mathematics Education*, 24, 233-254.

# PRESERVICE SECONDARY MATHEMATICS TEACHERS' INTERPRETATIONS OF MATHEMATICAL PROOF

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The purpose of this study was to examine the nature of preservice secondary mathematics teachers' understanding of mathematical proof and to examine their expectations of mathematical proof for students. Their understanding and expectations were characterized along four levels of proof, representing an hierarchical progression from inductive toward deductive generalizations. The findings suggest that several of the preservice teachers' understanding differed from what might be expected from prospective secondary mathematics teachers. Additionally, their expectations for their future students' understanding of mathematical proof were primarily characterized at the lowest level.

Proof holds a central role in the discipline of mathematics and is beginning to play a larger role in the secondary mathematics curriculum (outside the realm of geometry). In fact, teachers are being called upon to provide *all* students with rich opportunities and experiences with mathematical proof. Consequently, many mathematics educators have been re-examining the nature of mathematical proof in the secondary curriculum, and as a result, there has been a shift in emphasis, away from what has often been perceived as an over-reliance on rigorous proofs toward a conception of proof as convincing argument (Hanna, 1990). The classroom practice aligned with this conception of proof places an emphasis on developing mathematical reasoning through the social interactions occurring within the classroom community—interactions which provide an opportunity for students to "make conjectures and present solutions; explore examples and counterexamples to investigate a conjecture; try to convince themselves and one another of the validity of particular representations, solutions, conjectures, and answers; [and] rely on mathematical evidence and argument to determine validity" (NCTM, 1991, p. 45). In fact, Balacheff (1991) contends that "social interaction has been the real engine which leads the students to an awareness of the need for proofs, forcing them to justify themselves or to elicit the rationality of the decision to be taken" (p. 95).

Accordingly, teachers play a crucial role in promoting "the establishment of a classroom mathematics community in which mathematical validation and understanding are seen as appropriate and important foci" (Simon & Blume, 1996, p. 9). Moreover, as representatives of the mathematics community, teachers have the primary responsibility for establishing and negotiating what counts as an acceptable mathematical explanation and justification within the classroom community (Yackel & Cobb, 1996). Consequently, teachers' understanding of mathematical proof influences both the experiences they provide their students and the expectations of proof they hold for their students. The purpose of this study was to examine the nature of their interpretations by examining preservice teachers' understanding of mathematical proof and their expectations of mathematical proof for their future students.

## Conceptual Framework & Research Questions

In order to characterize the nature of the preservice secondary mathematics teachers' interpretations of mathematical proof, we utilized the four levels of Balacheff's (1991) taxonomy of proof: naive empiricism, crucial experiment, generic example, and thought experiment. Simon and Blume (1996) provide a succinct description of the four levels:

> At the first level the student concludes that an assertion is valid from a small number of cases. At the second level, the student deals more explicitly with the question of generalization by examining a case that is not very particular (e.g., choosing an extreme case). At the third level, the students develop arguments based on a 'generic example' (e.g., an example representative of a class of objects). At the fourth level, students begin to detach their explanations from particular examples and begin to move from practical to intellectual proofs (p. 8).

These levels represent an hierarchy that students are expected to progress through as their notions of mathematical proof develop, that is, their understanding of mathematical proof is "likely to proceed from inductive toward deductive and toward greater generality" (Simon & Blume, 1996, p. 9). Although these levels were originally used for categorizing secondary students' understanding of proof, we found Balacheff's distinctions provided a means for describing the nature of preservice teachers' understanding of proof and for examining their expectations for their students' understanding of mathematical proof.

The study was guided primarily by two questions: (1) What is the nature of preservice secondary mathematics teachers' understandings of mathematical proof? and (2) What expectations of proof do preservice secondary mathematics teachers hold for students who have not yet been exposed to formal deductive proofs?

## Methods

### Context

The participants were preservice secondary mathematics teachers (n = 9) who were in the process of completing or had previously completed an undergraduate degree (or its equivalent) in mathematics. The participants were enrolled in a semester-long, reform-based mathematics methods course. A major portion of each methods class was spent in the context of solving mathematics problems—the students attempted to solve each problem individually and/or in small groups, and the mathematics task, concepts and skills then provided the context for discussing a variety of pedagogical issues. In addition, the students were frequently assigned homework problems whose solutions were then discussed during the following class. Central to both the in-class and homework discussions were the mathematical justification students provided as they presented their solutions to the mathematics problems. This format not only provided a more dynamic examination of their notions of mathematical proof and its role in the teaching of mathematics, but it also promoted reflection on their own interpretations of mathematical proof.

### Sample Problems

The following two problems are representative of the types of problems that were assigned: (1) Given line k and two points A and B on the same side of the line, find point X on k so that the path A-X-B is as small as possible. Present an argument justifying your solution (L. Sowder, personal communication, Spring 1993); (2) Given a circle with center F, and a point inside the circle H, find a chord that passes through the point and produces the largest product of the two resulting segments. Present an argument justifying your solution (Cooney, 1988).

### Data Sources and Analysis

Data sources included video-tapes of class discussions, student write-ups of homework problems, and student interviews. The interviews focused on the students' descriptions of their solutions to homework problems and on their expectations for secondary school (pre-geometry) students' solutions on similar problems. Data—students' homework solutions, class discussions, and interview responses—were analyzed using Balacheff's four levels of proof. Each student's solution was categorized according to the level that best represented the student's response.

## Results

*What is the nature of preservice secondary mathematics teachers' understandings of mathematical proof?* We briefly describe the preservice teachers' solutions for the first problem and characterize their solutions in terms of Balacheff's four levels of proof.[1] Although it would be reasonable to expect all of the preservice teachers to present proofs at the fourth level in Balacheff's taxonomy, we were surprised to find two teachers who did not fall into this category. For this reason, we describe in greater detail the proofs these two teachers presented.

In solving the first problem, three different solution methods were utilized: a geometric approach, a calculus approach, and an empirical approach. Five of the preservice teachers employed a geometric approach in which they attempted to prove their solution deductively. Although these teachers might be classified as being at the fourth level (i.e., thought experiment), their proofs differed significantly. Only two of these preservice teachers provided a proof that we considered mathematically viable; the other three made an assumption regarding the correct location for the point X and then based their proofs on this assumption (the assumption in fact also required proof). The calculus approach was used by two of the preservice teachers in proving their solutions—proofs representative of the thought experiment level. They placed point X on line k, used the distance formula to find the length of A-X-B, and then set the derivative equal to zero in order to find the location of point X that produced the minimum distance.

The remaining two preservice teachers presented empirical arguments as justification for their solutions. One teacher decided initially to test three different possible locations for point X. She placed one test point, $X_1$, on line k such that the two angles formed by line k and the lines $AX_1$ and $BX_1$ were congruent. Next, she placed two additional test points on line k, one on each side of the first test point. She then measured the lengths of the three paths formed in order to determine the shortest path. At this point, she concluded that the first test point location for X produced the shortest path. Rather than repeating the previous steps with different placements of A and B, she decided to "formulate a more mathematical generalization about what is going on." Her mathematical generalization began with recalling "from my past geometry experience, that the shortest distance between two points is a straight path." Interestingly though, she proceeded to draw a diagram showing the reflection of point A about line k, but rather than attempting to prove her previous conclusion deductively, she chose to again verify it empirically—she measured the path from the reflected point to B and compared it with the length of path A-X-B. Finding these lengths to be approximately equal, she concluded that this location for X produced the shortest path. Based strictly on the work she presented, it seems as though her recollection of a fact from geometry was enough to warrant this proof being considered as a more mathematical generalization than her initial "trial and error" proof. We felt that her proof might best be characterized at the crucial experiment level; she strategically examined three possible locations, specifically trying to narrow her choices down to one potential winner, and then verified this choice as the correct location. In other words, the question of generalizability was explicitly examined, a distinguishing feature Balacheff ascribes to this level.

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[1] The characterization of the preservice teachers' solutions for other problems included in the study were consistent with their solutions to this first problem.

The second teacher, also using an empirical approach, began by assuming that the points A and B were equidistant from line k, specified a value for the distance from these points to line k, and specified a value for the distance between points A and B. Her resulting diagram was a rectangle with line k as a base, points A and B at the vertices of the opposite base, and point X located on line k, at the vertex opposite point A. She proceeded to calculate, using trigonometry, the path length of A-X-B for different locations of X (she began with X at its initial location and moved it to its final location, the midpoint between A and B on line k). She displayed the results in a table and decided to then graph the data in order to get "a better feel for the location of X." From the tabular and graphical representations, she determined the location for X that produced the minimum path. It is interesting that in her conclusion, she generalizes the results from a specific example to all cases—justification characteristic of Balacheff's first level. However, it is possible that she selected her example to be representative of a class of objects (i.e., all cases in which points A and B are equidistant from line k), which would then be more characteristic of Balacheff's third level, generic example. Although based solely on her written response (she was not interviewed), it is difficult to determine how she was thinking about the example she used.

*What expectations of proof do preservice secondary mathematics teachers hold for students who have not been exposed to formal deductive proofs?* Six of the seven preservice teachers interviewed expected their students to offer proofs at the naive empiricism level. The following description typifies the proof they expected their students would present for the second problem. The students would draw a circle with center F and a point H inside the circle. Next they would draw several chords that passed through point H. They would then measure the lengths of the two segments formed by each chord, find the product for each pair of segments, and compare the resulting products. Finally, taking into account any measurement error, the students would then conclude that all chords through the given point would produce equal segment products. The preservice teachers did not expect their students to go beyond this point, although several mentioned that they would at least point out to their students that they had only measured a small number of all the possible chords, and therefore their conclusion was tenable at best. Interestingly, only one preservice teacher expected students to demonstrate a more sophisticated understanding of proof—a proof representative of Balacheff's crucial experiment. This teacher expected his students to check several different circles, several different point locations, and chords that included the longest and the shortest (i.e., the shortest they could reasonably draw and accurately measure) chords. It is important to distinguish the difference between the preservice teachers' expectations for the mathematical thinking of their students. In the former situation, the teachers expected students to be satisfied with their justification based on a small number of specific cases, whereas in the latter situation, the teacher expected students to feel the need to test particular cases (e.g., longest and shortest cases) and test different circles and point locations. This teacher wanted students to recognize the need to check more than a few cases, cases that aren't typical, and cases that the students felt would really test the validity of their assertion. It is also interesting to note that none of the preservice teachers expected their students to produce proofs at the generic example level, the highest level students without formal deductive proof experience might attain.

## Concluding Remarks

The results suggest that several of the preservice teachers' interpretations of mathematical proof differed from what the mathematics community would consider as mathematically acceptable. Further, most of the preservice teachers' expectations of proof for their students was limited. Two explanations for these findings seem feasible: (1) the preservice teachers possess an inadequate understanding of mathematical proof or (2) the preservice teachers have a different notion of what is an acceptable mathematical proof outside of the formal setting of their university mathematics classes. In either case however, the end result—student understanding of proof and its use in mathematics—is problematic. This is particularly true for students who lack experience with formal deductive proof. As Martin and Harel (1989) warn, "if teachers lead their students to believe that a few well-chosen examples constitute proof, it is natural to expect the idea of proof in high school geometry and other courses will be difficult for the students" (p. 42). Consequently, if teachers are to support the development of their students' understanding of proof, it is important that teachers not only have a robust understanding of proof, but that they are also afforded opportunities to "explore, develop mathematical arguments, conjecture, validate possible solutions, and identify connections among mathematical ideas" (NCTM, 1991, p. 128). Such opportunities not only provide a more dynamic examination of the nature of mathematical proof in the secondary curriculum, but they also engender discussion concerning expectations of proof for students.

## References

Balacheff, N. (1991). Treatment of refutations: Aspects of the complexity of a constructivist approach to mathematics learning. In E. von Glasersfeld (Ed.), *Radical constructivism in mathematics education* (pp. 89-110). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Cooney, T. (1988). The issue of reform: What have we learned from yesteryear? *Mathematics Teacher, 81*, 352-363.

Hanna, G. (1990). Some pedagogical aspects of proof. *Interchange, 21* (1), 6-13.

Martin, W. G., & Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education, 20* (1), 41-51.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

Simon, M., & Blume, G. (1996). Justification in the mathematics classroom: A study of prospective elementary teachers. *Journal of Mathematical Behavior, 15* (1), 3-31.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education, 27* (4), 458-477.

# WHY DO WE INVERT AND MULTIPLY? ELEMENTARY TEACHERS' STRUGGLE TO CONCEPTUALIZE DIVISION OF FRACTIONS

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To study elementary teachers' construction of understanding of division of fractions, we conducted a teaching experiment with a class of 12 teachers and case studies with three of the teachers. Initially, the teachers were able to create meaningful division situations of fractions and solve them using the invert-and-multiply (I&M) algorithm, but they could not make sense of the algorithm. During instruction, teachers solved tasks in a computer microworld which led to their construction of fractions as co-measure units, a process that had previously been observed in children. The teachers observed that the result of using the co-measure units was consistent with the algorithmic answer. However, they were not able to pose and solve tasks in the "opposite of multiplication" context, which would allow them to relate their mental operations with the algorithm. The study demonstrates how a teacher educator can use research on stages of children's learning about fractions to organize observations of teachers' knowledge and to devise situations that promote teachers' understanding.

This paper addresses the problem of how elementary teachers might construct conceptual structures and operations necessary for relational knowledge (Skemp, 1987) of the division of fractions algorithm, "invert-and-multiply" (I&M). In particular, it demonstrates how conceptions identified in research on children's learning can contribute to advancing and studying teachers' development. Addressing this problem is important because many elementary teachers cannot facilitate children's construction of such knowledge because they have not constructed it themselves (Post, Harel, Behr, & Lesh, 1991). As a teacher in our study put it, teaching division of fractions is characterized by the rhyme: "Invert and multiply, and don't ask me why." The paper outlines the constructivist framework and method used to study the problem, presents analysis of the teachers' work, and discusses the significance of the teachers' evolving understandings.

## A Constructivist Theoretical Framework

We view human learning in the context of scheme theory (von Glasersfeld, 1989), as the continual processes of accommodating established conceptual structures and operations to neutralize perturbations. In this context, a teacher can promote learning of specific mathematical knowledge by engaging the learners in solving tasks designed to create specific mathematical perturbations in the learners. To do so, the teacher needs to infer the learners' conceptions as they solve, or fail to solve the tasks, and to generate conjectures as to how they might modify these conceptions. Simon (1995) refers to such conjectures as a hypothetical learning trajectory (HLT). Since research on teachers' fraction knowledge (Ball, 1990; Lehrer & Franke, 1992) provided us with no hypotheses for inferring teachers' established schemes or for implementing previously tested HLT's, we adapted findings from research on children's fraction learning.

Tzur (1996) proposes the *iterative fraction scheme* as a transformation in children's thinking about fraction units, from one part out of several equal parts contained in a partitioned ONE to a relation between the part and the ONE. For a child with the latter knowledge, the word "one-eleventh" symbolizes a fraction unit that stands in 1-to-11 relation with the ONE, a relation the child anticipates without needing to iterate the fraction unit 11 times. Tzur (1995) found that a child can then construct the *distributive partitioning scheme* by abstracting, in 3 sub-stages, the results of the operation of partitioning parts (*recursive partitioning*). First, a child makes sense of partitioning, say, 1/4 into 5 parts, by distributing 5 parts across each of the 4 fourths, finding the total number of parts in the ONE (20), and translating it into a fraction unit (1/20). Next, the child abstracts the multiplicative relation between two recursive partitions, and can anticipate the size of the resultant fraction without executing the distribution. Then, the child generalizes the latter abstraction to several recursive partitions, which is a case of the fundamental operation of splitting (Confrey & Smith, 1995).

Olive (1993) proposes the construction of *co-measure units and scheme* as the next advancement in the child's fraction knowledge. In this stage, the child abstracts and *reverses* the relationship between the recursive partitioning operation and its resultant sub-units, thus being able to *anticipate, without executing, the result of using this operation* to produce fraction units. For example, the child conceives 1/30 as a fraction unit that could produce or relate 1/2, 1/5, and 1/6. Both Olive and Tzur hypothesize that those schemes underlie understanding of arithmetical operations such as: (a) finding common denominators, (b) adding unlike-denominator fractions, (c) simplifying fractions, and (d) multiplying/dividing fractions.

## Methodology

We studied the development of 12 elementary school teachers who participated in a master's course taught by the first author, in which they used the Sticks computer microworld. In this microworld the user can draw linear objects ("sticks") on the screen, place vertical marks at any point on a stick (and erase them), copy a stick several times, join sticks, duplicate (repeat) sticks, partition sticks (or parts) into a number (2-99) of equal parts, break marked/partitioned sticks into pieces, cover/uncover an area on the screen, designate a stick as the measuring unit of ONE by copying it into the "Ruler" and then measure any stick on the screen, label sticks with fraction numerals, and disembedded parts from a marked or partitioned stick without changing the original. Thus, Sticks provides learners with an environment in which they can actively explore, or demonstrate understandings of fractional relationships between sizes of one-dimensional objects.

We videotaped every lesson, with one camera recording the entire group activities and the second focusing on case study teachers. The second author conducted case studies of three of the teachers, including four audiotaped interviews with each teacher, collection of written work, and observation of the teachers in their classrooms. Finally, we transcribed the tapes (video and audio) and analyzed the relevant segments line-by-line.

## Analysis

After considerable work on division problems with whole numbers, the teachers' solutions indicated that they understood division in 3 contexts—partitioning, measurement, and "opposite of multiplication" (e.g., solving 2((= 6 by 6 ÷ 2). Understanding the latter context was important since it underlies a possible explanation of the I&M algorithm. For example, one can solve the problem "I'm thinking of a stick that the 3/4 is 2/5 of it" by: (a) partitioning each of the 3 fourths into 2 parts to create 1/5 of the desired outcome, hence dividing it into 3/8, and (b) iterating this sub-unit 5 times to complete the process while producing *15/8 of the original ONE*. In this solution, one multiplies 3/4 by 5/2 (the inverted reciprocal of 2/5) in two steps—first dividing by 2 and then multiplying by 5.

In class 12 (4-3-96), we discussed the teachers' solution to a problem the instructor (I) posed the previous week in the context of "opposite of multiplication":

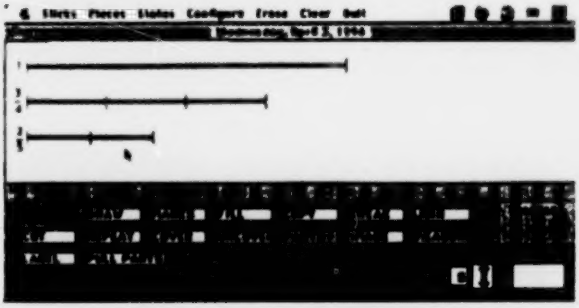
I: I thought of a stick that yours was 5/8 of it. What exactly did you do with your 5/8 to make mine?

P: We divided the 5/8 by 5/8.

I: (Emphasizes that he meant what Sticks actions did they use, and reminds them that they first divided the 5/8 into 5, then took 8 of those. Next, he poses the task which the students would solve in small groups): The question is why do we invert and multiply? And the task is, use one or more aspects of division, that is, partitioning, measur-

> ing, or "opposite of multiplication," and Sticks microworld, and the fractions 3/4 and 2/5, to come up with a problem or a situation, that is a division problem. . . . Make something that is similar to mine, or a different problem.

Unlike in Ball's (1990) study, the teachers could easily find a measurement situation such as "How many scarves can we possibly make if it takes 2/5 of a yard to make one scarf and we have 3/4 of a yard of cloth?" and solve it using the I&M algorithm. They also had no problem making a 3/4-stick and a 2/5-stick to represent the two quantities, and rephrasing the task as "having to find how many times 2/5 goes into 3/4" (see Figure 1). Yet, only one student (D) was able to solve the problem. As D attested in her reflective journal and in class, she used the numbers obtained by the algorithm (15/8) to find a way of creating common denominators between the two fractions, a way she termed "working from the solution backward."



In conceptual terms, D recursively partitioned each of the three-fourths into 5 parts (hence 15/20) and each of the two-fifths into 4 parts (hence 8/20), then used the precise co-measure unit (1/20)to figure out that they could possibly make 15/8 scarves (or 1 and 7/8). However, in spite of initiating and solving many simpler problems, D did not know why her method worked:

D: (in class 14, 4-17-96): It [my method] works, but I'm not really [sure why].

I: So is this only an accident or maybe there is something in it?

D: I think there's something in it, I've tried it on a lot of simpler problems ...

Our analysis of class 14 focused on our puzzlement as to why most teachers were not able to follow D's activities to figure out "how many times 2/5 goes into 3/4," and why the teachers reverted to a measurement division context instead of following the instructor's "opposite of multiplication" example. Building on our understanding of stages in children's conceptualization, we felt it was necessary to examine and advance the teachers' establishment of an anticipatory relationship between the results of two consecutive partitions and the second step (multiply the fractions) of the I&M algorithm.

In class 15 the instructor first posed tasks such as "Could you figure out how much is 3/4 *of* 1/5 *of* the ONE?" to foster teachers' conceptual reorganization of: multiplication of fractions, partitioning parts, and the units of units conception. The teachers partitioned the ONE into 5 parts and 1/5 into 4 parts (1/20), disembedded 3 such parts, and explained that it was 3/20. Yet, they had great difficulty when trying to solve the reciprocal task (1/5 *of* 3/4). Unlike the first task in which the teachers could regard the single fifth as a unit in itself, in the reciprocal task they worked with 3 parts (1/4 each), which required co-ordinating the operation of partitioning parts and the operation of uniting (composing) 3 of the resultant sub-units. Their difficulty in anticipating the results of partitioning parts was seen in D's solution. She partitioned each fourth into 5 parts, disembedded one such part from each fourth, and joined them, but instead of co-ordinating these activities to figure out how much this part was of the whole (3/20) she reverted to the established operation of iterating that part (6 and 2/3 times), which left her with a perturbing, meaningless number. In turn, this perturbation led her to reflect on the recursive partitioning acts, which led to realizing that the resultant sub-unit was 1/20 of the ONE, and that they had 3 such parts, hence 3/20. This realization brought about an "AHA" experience—the ability to grasp why the result of both tasks was the same as the result obtained via the multiplication algorithm.

Class 15 was the end of the semester, but the teachers indicated a high interest in additional work on division of fractions. Thus, we added class 16 (5-1-96) during the final exams week. Our analysis of the teachers' work in class 15 suggested that they needed to advance their anticipation of the result of partitioning parts and to use it to solve problems that require the creation of co-measure units. To this end, the instructor first engaged them in solving tasks such as figuring out the result of partitioning the ONE into 3 parts, then 1/3 into 4 parts, then 1/12 into 3 parts, etc. The teachers' work (in pairs) indicated that they were able to anticipate the result of a potential (next) partitioning act prior to executing it. For example, they knew that if they were to partition 1/36 into 4 parts, each of the resultant parts would be 1/144 of the ONE because the ONE can potentially consist of 144 such parts.

Then, the instructor asked them to create 1/5 of the ONE from a piece which was 1/4 of the ONE. To do so, they partitioned (in an anticipatory way) the 1/4 into 5 parts, then disembedded and iterated this co-measure unit (1/5 of 1/4) four times to make 1/5 of the ONE. This solution indicated *the creation of 1/20 as a co-measure unit for 1/4 and 1/5 of the ONE*. Next, they worked on addition problems such as 2/6+1/7. They solved this task by partitioning each sixth into 7 parts and each seventh into 6 parts to create 1/42 as the co-measure unit, then adding both to find the result (14/42 + 6/42 = 20/42). In all, the teachers' solutions of those tasks indicated to the instructor an anticipatory and flexible co-ordination of partitioning parts and uniting the resultant sub-units (distributive scheme). Thus, he asked them to work on their scarves (division) problem.

After a short time, all groups' solutions indicated how they essentially anticipated the need for creating a co-measure unit (1/20) to figure out how many times the 2/5-stick goes into the 3/4-stick. They seemed to understand: (a) the change of the measurement unit from one yard to one scarf (2/5 of a yard), (b) that this process can be generalized for other fractions, and (c) that the results are consistent with those obtained via the I&M algorithm. However, as D's concluding response to the instructor's question "How is this related to invert and multiply?" indicated, they did not yet understand why: "Well, I don't know, it just works with the numbers."

## Discussion

After several weeks of struggling with the perturbation of "Why do we invert and multiply?", including the additional class during final exam week, the teachers appeared to have constructed relational knowledge that: (a) enabled their evolving understanding of multiplicative operations with fractions and (b) *could lead* to making sense of the I&M algorithm. The results imply that teachers need to construct similar conceptual stages found in children's learning, using similar tasks, before they can make sense of the accepted mathematical algorithms of adding and multiplying fractions. In this sense, adapting conceptual stages (and tasks) from research with children to advance teachers' knowledge of the I&M algorithm substantiated and extended these conceptual stages, and served as the basis for the instructor's hypothetical learning trajectory.

The teachers' learning included several conceptual transformations resulting from a long struggle to make sense of the perturbing, "magical" I&M algorithm. Yet, their work was only a step in constructing the conceptual structures and operations necessary for understanding the algorithm, because creating 15 sub-units was based on dividing three-fourths by 5, not on multiplying them as implied by the algorithm. The teachers' solutions indicated an anticipation of a multiplicative pattern in the results of the operation—every time they partition the parts reciprocally (e.g., each 1/4 into 5 and each 1/5 into 4) they can extract the numbers obtained by the algorithm. However, it would require more research to articulate how they may make a deeper sense of the I&M algorithm, e.g., by solving problems in the "opposite of multiplication" context.

217

## References

Ball, D. L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education, 21*(2), 132-144.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(1), 66-86.

Lehrer, R., & Franke, M. (1992). Applying personal construct psychology to the study of teachers' knowledge of fractions. *Journal for Research in Mathematics Education, 23*(3), 223-241.

Olive, J. (1993). *Children's construction of fractional schemes in computer microworlds, Part II: Constructing multiplicative operations with fractions.* A paper presented at the 71st Annual Meeting of the National Council of Teachers of Mathematics, Seattle, WA.

Post, T. R., Harel, G., Behr, M. J., & Lesh, R. (1991). Intermediate teachers' knowledge of rational number concepts. In E. Fennema, T. P. Carpenter, & S. J. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 177-198). Albany, NY: SUNY Press.

Simon, M. A. (1995). Elaborating models of mathematics teaching: A response to Steffe and D'Ambrosio. *Journal for Research in Mathematics Education, 26*(2), 160-162.

Skemp, R. (1987). Relational understanding and instrumental understanding. *Arithmetic Teacher, 26*(3), 9-15.

Tzur, R. (1995). *Interaction and children's fraction learning.* Unpublished doctoral dissertation, The University of Georgia, Athens.

Tzur, R. (1996). Children's interaction and fraction learning in a computer microworld: Construction of the iterative fraction scheme. *Proceedings of the Twentieth Annual Meeting of the International Group for the Psychology of Mathematics Education* (vol. 4, pp. 355-362). Valencia, Spain: University of Valencia.

Von Glasersfeld, E. (1989). Cognition, construction of knowledge, and teaching. *Synthese: A special issue of philosophy of science and education.*

# UNDERSTANDING MATHEMATICS CURRICULUM: TEACHERS MAKING SENSE OF PROBLEM SOLVING, COMMUNICATION, CONNECTIONS, AND REASONING

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Across North America, policy-makers believe that students will be able to achieve higher levels in problem solving (International Assessment of Education Progress, 1992; School Achievement Indicators Program, 1993; National Council of Teachers of Mathematics, 1989) if students have experience with mathematics curricula that integrates the processes of problem solving, reasoning, communication, and connections throughout content. Curricula are being developed to reflect this belief; however teachers' understanding of these processes will affect their actions in implementing these new curricula. This paper investigates teachers' pursuit of meaning of problem solving, communication, connections, and reasoning.

The data for this paper is taken from a larger study of teachers making sense of these mathematical processes and their translation of these processes into practice. Each teacher from one of the groups in the project was asked to write what he/she believed the meaning of these mathematical processes were and then through conversation in a group, develop a taken-as-shared meaning of these processes.

Teachers brought forth a world of significance about problem solving, reasoning, connections, and communication with others in their group. Each individual teacher's structure and definition of these words were being changed as they participated in negotiating meaning of these processes. From a narrative perspective, the individual teacher's story about these four mathematical processes was constantly being changed by his/her conversations and interactions.

## References

Council of Ministers of Education, Canada. (1993). *School achievement indicators program report on mathematics assessment.* Toronto, ON: Author.

Educational Testing Service. (1992). *The international assessment of educational progress performance assessment: An international experiment.* Princeton, NJ: Author.

Educational Testing Service. (1992). *The international assessment of educational progress learning mathematics.* Princeton, NJ: Author.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

# THE INFLUENCE OF ONE TEACHER'S KNOWLEDGE OF HER STUDENTS' MATHEMATICAL THINKING ON HER PRACTICE

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Teacher's knowledge of their students' mathematical thinking significantly influences teachers' beliefs and practices. This paper draws on a qualitative case study of one teacher as she attempted to realize a constructivist, reform-oriented approach to teaching mathematics in her second-grade classroom. The study suggests that helping teachers develop knowledge of how their students solve problems and think mathematically, is one of the most salient aspects in realizing reform efforts. Further this knowledge development was ongoing and occurred in the course of the teachers' interactions with students.

Excerpts from the case study will be presented to illustrate how one teacher, Mary, developed her knowledge of students' mathematical thinking in the course of her practice. Mary learned more about how her students thought mathematically and with this new knowledge she attempted to help students develop viable solutions to problems. In specific classroom interactions she might suggest that students use manipulatives to solve problems or have children explain their solution method in an attempt to help them reflect on their own thinking and resolve conflicts. By listening to these explanations it appeared, from the researchers perspective, that she had come to believe that what students said made sense to them, however strange it might seem to her. As consequence she changed her role from one who directed students to answers, to one who tried to understand and influence the process they used. Classroom interactions with students were the key influence for Mary to change her teaching practices. Her knowledge of her students mathematical thinking and her changes in her teaching were reflexively related.

A report on the CGI project corroborates the assertion that teachers' knowledge of their students' mathematical thinking is at the heart of teachers' learning. Knapp and Peterson (1995) concluded that, "[Teachers] reported learning mainly through their interactions with students and other teachers." Enhancing teachers practices by providing them with research-based knowledge may be a viable step, but it does not explain teachers' learning and change in practice. This study suggests that we must look at more than increasing the subject matter knowledge of teachers or providing them with pedagogical knowledge in order to realize reform-oriented classrooms. More appropriately, the mathematics education community might consider providing teachers with opportunities to learn about their students' mathematical thinking.

## References

Knapp, N., & Peterson, P. (1995). Teachers' Interpretations of "CGI" after four years: Meaning and practices. *Journal for Research in Mathematics Education, 26*(1), 40-65.

# PROCEDURAL AND CONCEPTUAL KNOWLEDGE IN ONE TEACHER'S EXPLANATIONS

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## Objectives

The objective of this paper is to examine the explanations that one knowledgeable teacher, Mr. Kantor, constructed when teaching topics related to eighth-grade algebraic multiplication in terms of procedural and conceptual knowledge (Hiebert & Lefevre, 1986).

## Data Collection and Procedures

Through a content analysis of the ten lessons of the textbook dealing with algebraic multiplication, about 41 mathematical ideas for which the textbook provides representations or explanations were identified. Since the textbook provides representations and or explanations, it was reasonable to assume that Mr. Kantor would construct explanations for some of these mathematical ideas. Mr. Kantor's classroom instruction was videotaped to examine the degree of conceptual and procedural knowledge in his explanations.

## Data Analysis and Results

I attempted to categorize Mr. Kantor's explanations into four categories: instrumental-procedural, procedural-conceptual, conceptual-procedural, and conceptual-and-procedural rich explanations. Mr. Kantor only constructed explanations for 18 of the 41 possible mathematical ideas. All explanations involved some elements of procedural and conceptual knowledge with procedural knowledge emphasized in ten explanations and conceptual knowledge emphasized in 8 explanations. However, the degree of both conceptual and procedural knowledge could have been much stronger.

## Conclusion

The findings suggest that knowledgeable teachers can use their knowledge to construct explanations involving both conceptual and procedural knowledge. However, the relationship between teachers' knowledge and use of that knowledge during classroom instruction to construct explanations involving strong elements of both procedural and conceptual knowledge is far from being a linear relationship.

## References

Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge in mathematics: An introductory analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1-27). Hillsdale, NJ: Lawrence Erlbaum.

# EXPERIENCED SECONDARY TEACHERS' KNOWLEDGE OF DIVISION AND ITS RELATIONSHIP TO MULTIPLICATION: TWO CASE STUDIES

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The main objective of this paper is to examine in detail how two experienced eighth grade mathematics teachers, Mr. Kantor and Mr. Taylor, understand the concept of division and its relationship to multiplication in four contexts: (a) concept of division, (b) definition of division as $a \div b = \frac{1}{b}, b \neq 0$, (c) division by zero, and (d) the rules of signs.

The teachers were asked to create, when appropriate, a story problem, a picture, a symbolic representation, and a proof of the main mathematical ideas related to division and its relationship to multiplication using audiotaped interviews and questionnaires. Teachers' conceptions of division included both partitive division and quotitive division. As an example, Mr. Kantor constructed the problem: "you have five loaves of bread and give half loaf to each family. How many families can you feed?" Both teachers represented the symbolic definition of division as $a \div b = \frac{1}{b}, b \neq 0$. Both teachers constructed correct story problem representations to illustrate why $a \div b = \frac{1}{b}, b \neq 0$. For example, Mr. Taylor, illustrated why $21 \div \frac{3}{4} = 21 \cdot \frac{4}{3}$ using the word problem "I got twenty one ounces of some chemical. I need three fourths for each experiment . . . how many experiments can I do? . . . for one ounce you are gonna carry out one and a third experiment. So, in other words, four thirds experiments are going to come from one ounce. . . . You have twenty-one of those ounces, (so) $\frac{4}{3}$ times . . . 21 gives you the total number of experiments." Both teachers stated that division by zero was impossible or undefined. However, Mr. Taylor was unsure about the case $0 \div 0$.

Mr. Kantor and Mr. Taylor knew representations about the concept of division and its relationship to multiplication. From a teaching perspective, the question of the impact of those teachers' knowledge on their classroom instruction remains open. We address this question in the sequel to this paper (Contreras, 1997).

## References

Contreras, J. (1997). *Learning to teach algebraic division for understanding: A comparison and contrast between two experienced teachers.* Paper accepted for presentation at PME-NA XIX.

# A COMPARISON OF STUDENT BELIEFS: HIGH SCHOOL PRE-ALGEBRA AND COLLEGE DEVELOPMENTAL MATH

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Many studies have shown that programs at colleges and universities which address the affective factors, as well as the cognitive, in teaching students have been successful with students who have had difficulty learning traditional mathematics in high school. At the same time, *Curriculum and Evaluation Standards for School Mathematics* (National Council of Teachers of Mathematics, 1989), has encouraged mathematics educators at all levels to increase their awareness of affective factors in the learning process and not focus solely on cognitive factors.

This study investigated the beliefs of two groups: remedial math students at a rural high school, and developmental math students at a state university. Results of a Chi-Square analysis of free response questions regarding self-beliefs and beliefs about mathematics indicated that both groups (high school and college) had relatively the same beliefs about their mathematical ability. Consequently, vertical alignment of the affective curriculum could be explored. It could be suggested that affective techniques used by colleges which are successful in remediating students could be used by teachers in high school to address students' affective needs. If high school teachers could use techniques to meet the affective needs of their students as well as maintain cognitive standards, then students could achieve success in mathematics earlier in their schooling, thus lessening the need for remediation in college.

# TECHNOLOGY

# *SHAPE MAKERS*: A COMPUTER MICROWORLD FOR PROMOTING DYNAMIC IMAGERY IN SUPPORT OF GEOMETRIC REASONING

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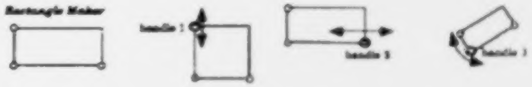
This paper describes research on the development of students' pre-proof geometric thinking in a specially-designed computer microworld. It focuses on students' transitions from visual to property-based thinking, and from property-based thought to thinking that utilizes inference to relate and organize both properties and classes of shapes. It analyzes episodes of pairs of students who were working at computers during classroom instruction conducted by the students' regular classroom teacher. The theoretical framework integrates two perspectives—that of the van Hiele hierarchy, and that of mental models—all within a constructivist paradigm. First, we briefly describe the microworld; second, we summarize our theoretical framework; and, finally, we give several examples of our preliminary analysis of data.[1]

## The *Shape Makers* Computer Microworld

The *Shape Makers* computer microworld (Battista, in press) was designed to promote in students the development of dynamic mental models that they can use for reasoning about geometric shapes. In the microworld, each class of common quadrilaterals and triangles has a "Shape Maker," a *Geometer's Sketchpad* construction that can be dynamically transformed in various ways, but only to produce different shapes in the class. For instance, the Rectangle Maker can be used to make any desired rectangle that fits on the computer screen, no matter what its shape, size, or orientation—but only rectangles. It is manipulated by using the mouse to drag its *handles*—small circles that appear at its vertices—to make it taller, wider, or change its orientation and size.



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[1] Partial support for this work was provided by the National Science Foundation under grant RED 8954664. The opinions expressed, however, are those of the authors and do not necessarily reflect the views of that foundation.

## The van Hiele Levels of Geometric Thinking

According to van Hiele, students progress through several qualitatively different levels of geometric thinking (Clements & Battista, 1992). Geometric thought begins at the Gestalt-like visual level in which students identify and operate on shapes and other geometric configurations according to their appearance. It progresses to the level of description and analysis in which students recognize and can characterize shapes by their mathematical properties. At the next level, students' geometric thinking becomes abstract and relational as they see that one property can signal other properties, define classes of shapes, distinguish between necessary and sufficient conditions for classes of shapes, understand and provide "locally" logical arguments for assertions, and hierarchically classify shapes. At the fourth level, students can comprehend and create formal geometric proofs, and at the fifth, students can compare axiomatic systems.

## Geometric Reasoning and Mental Models

Research suggests that the reasoning exhibited by students during the first three van Hiele levels can be accomplished with mental models (Battista, 1994). A mental model is an analog mental version of a situation whose structure is isomorphic to the perceived structure of the situation that it represents (Johnson-Laird, 1983). Individuals reason about a situation by activating mental models that enable them to simulate interactions within the situation so that they can explore possible scenarios and solutions to problems. When using a mental model to reason about a situation, a person can mentally move around, move on or into, combine, and transform objects, as well as perform other operations like those that can be performed on objects in the physical world. "The behavior of objects in the model is similar to the behavior of objects that they represent, and inferences are based on observing the effects of the operations" (Greeno, 1991, p. 178). Furthermore, individuals use of mental models is constrained by their knowledge and beliefs. That is, much of what happens when we form and manipulate a mental model reflects our underlying knowledge and beliefs about what would happen if we were dealing with the objects they represent.

### Mental Models and Understanding

According to the mental model view of the mind, individuals understand or make sense of a situation or a set of connected verbal propositions describing a situation when they can construct or activate a previously constructed mental model to represent the situation (Johnson-Laird, 1983). As mental models are constructed through a recursive process of successive abstraction, the results obtained with them have different degrees of representational power. At first, the models may be employed only in the presence of the physical phenomena they represent. Later, they can be activated in the absence of these phenomena. And still later, they can be used to simulate never-performed actions on phenomena. The power of mental model-based reasoning increases as the level of abstraction and degree of generality of the models increase.

## Sample Analyses

We will now present several examples that illustrate the type of data and focus of analysis of our research. All examples are from fifth-grade classrooms with 10- and 11-year-old students.

### Episode 1

In his initial manipulations of several Shape Makers, MI commented:

MI: [On the Kite Maker] If I pull one end out, the other end goes out.

[After trying to make a non-square rectangle with the Square Maker and concluding that it couldn't be done] The Square [Maker] would only get bigger and twist around—so it can't make a rectangle.

MI discovered constraints for the Shape Makers, but these constraints were not typical mathematical properties—that is, they were not explicit relationships between parts of shapes.

### Episode 2

Three students were investigating the Square Maker.

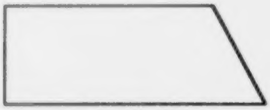
MT: I think maybe you could have made a rectangle.

JD: No; because when you change one side, they all change.

ER: All the sides are equal.

MT, JD, and ER have abstracted different things from their Shape Maker manipulations. MT noticed the visual similarity between squares and rectangles, causing him to conjecture that the Square Maker could make a rectangle. JD abstracted an action-based property—when one side changes length, all sides change. ER conceptualized a traditional mathematical property. In essence, JD and ER referred to the same constraint on the Square Maker. But ER described this constraint abstractly in terms of a relationship between measurements of the Shape Maker's sides.

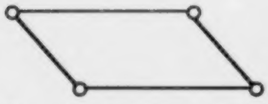
**Episode 3**. Three students were considering whether the Parallelogram Maker could be used to make the trapezoidal target figure at the right. Their knowledge of the Parallelogram Maker was



Target Figure

insufficient to predict that this was impossible. However, as they manipulated the Parallelogram Maker, one of the students discovered something about it that enabled her to solve the problem.

ST: [Pointing to the non-horizontal sides in the Parallelogram Maker] No, it won't work. See this one and this one stay the same, you know, together. If you push this one [side] out, this one [the opposite side] goes out...This side moves along with this side.

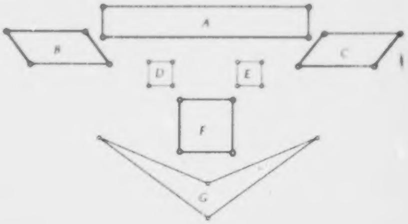


Parallelogram Maker

As ST manipulated the Parallelogram Maker in her attempts to make the target figure, she detected a pattern or regularity in its movement. After she abstracted and incorporated this movement pattern into her mental model for the Parallelogram Maker, she was able to infer that the target figure was impossible to make. By using the well-developed mental operations she had available for reasoning about physical objects and images, ST made a discovery that can, with further elaboration, form the basis for making sense of the formal mathematical property "in a parallelogram, opposites sides are parallel and congruent."

### Episode 4: Analysis Leads to Properties

NL is using the quadrilateral Shape Makers to make the design at the right. One of the researchers is observing and asking questions as NL tries to make shape C with the Rhombus Maker.



NL: I don't think that this is going to work...When I try to fit it on the shape, and I try to make it bigger or smaller, the whole thing moves. It will never get exactly the right size...

Res: When you tried to fit the Rhombus Maker on C, did you notice anything about shape C or the Rhombus Maker?

NL: The Rhombus Maker could make the same shape pretty much, but if you tried to make it small enough to fit on C, it would make the whole thing smaller, or it would move the shape down. And when you tried to move it up to make it smaller, it would make the whole shape move up.

Res: You said the Rhombus Maker could make the same shape, what do you mean by that?

NL: It could make this shape, the one with 2 diagonal [oblique] sides and 2 straight [horizontal] sides that are parallel. It could have been almost that shape [C] and it got so close I thought it was that shape. See it could make the same shape as that [shape C]. [NL manipulates the Rhombus Maker, making different angles, trying to get it to make an elongated parallelogram.]

Oh, I see why it didn't work, because the 4 sides are even and this [shape C] is more of a rectangle.

Res: How did you discover that?

NL: All you can do is just move it from side to side and up. But you can't get it to make a rectangle. When you move it this way it is a square and you can't move it up to make a rectangle. And when you move this [other handle] it just gets a bigger square.

Res: So what made you notice that?

NL: Well I was just thinking about it [the Rhombus Maker]. If it was the same shape then there is no reason it couldn't fit onto C. But I saw when I was playing with it to see how you could move it, that whenever I made it bigger or smaller, it was always like a square, but sometimes it would be leaning up, but the sides are always equal.

This episode clearly shows how a student's manipulation of a Shape Maker and resultant reflection on that manipulation can enable the student to move from thinking holistically to thinking about interrelationships between a shape's parts, that is, about its mathematical properties. Indeed, NL began the episode thinking about the Rhombus Maker and shapes holistically, saying that she was trying to make the Rhombus Maker "bigger or smaller," but "the whole thing moves."

The fact that NL could not make the non-equilateral parallelogram with the Rhombus Maker caused her to reevaluate her conception of the Rhombus Maker and reformulate it in terms of a mental model that explicitly incorporated her newly discovered mathematical property. Originally, because her model was not constrained by the property "all sides equal," her mental simulations of changing the shape of the Rhombus Maker included transforming it into non-equilateral parallelograms. Her subsequent attempts to manipulate the actual Rhombus Maker into a non-equilateral parallelogram tested her model, showing her that it was not viable. As she continued to analyze why the Rhombus Maker would not make the parallelogram—why it would not elongate—her attention shifted to the possibilities for changing its side lengths. This new focus of attention enabled her to abstract the regularity that all sides are equal.

**Episode 5: Shape Makers As Representations of Classes of Shapes.**

One of our major hypotheses is that a Shape Maker can become for students a "concrete" embodiment for the *class* of shapes given by its name. The Rectangle Maker, for example, can represent the class of all rectangles because the properties embodied by it are exactly those properties that all rectangles have. As the example below illustrates, this Shape Maker to shape-class correspondence enabled students to interrelate classes hierarchically, a characteristic of van Hiele level 3.

BE: A square is a rectangle, but a rectangle is not a square.

MA: I agree. The Rectangle Maker can make a square, but the Square Maker cannot make all rectangles.

SO: Every shape made by the Square Maker can be made by the Rectangle Maker because a square is a rectangle.

MA's statement shows how he was using his knowledge of the Shape Makers to justify BE's claim that squares are rectangles. Because MA and SO took the Shape Makers as representations of classes of shapes, they could reflect on their mental models of the Shape Makers and draw conclusions about properties of, and interrelationships between, classes of shapes.

**Episode 6: Implication**

Another example of level 3 thinking is demonstrated by the following instances in which students infer one property of a shape from another. The students are explaining why the Right Triangle Maker can't make a triangle with all sides equal.

NL: The line across from the right angle, it has to be longer because if it was shorter, then the two lines that form the right angle, it wouldn't be a right angle, it would be less than 90°.

TM: The diagonal line [hypotenuse] is longer, because if the two lines are there and they are both straight, it will take longer for the diagonal to get to each point—to connect to each point.

In these episodes, NL and TM reasoned by activating their mental models of various triangle Shape Makers. They made correct inferences not by logically deducing theorems from axioms, but by generating possibilities through simulations with these models. In fact, the mental model theory of inference assumes that individuals construct or activate a set of models, or a single dynamic transformable model, to represent premises and generate as well as reflect on possible implications (Johnson-Laird, 1983). The theory also postulates that individuals' inferencing passes through several levels of sophistication (Markovits, 1993). At first, individuals pay little attention to the exhaustiveness of their model-based search for possibilities relevant to the implication. Their mental models incorporate possibilities that have been directly experienced and can thus be accessed by reference to episodic memory (i.e. memory of personally experienced events). For instance, in their predictions, NL and TM were reflecting on mental models they had abstracted from their previous actions with the Shape Makers. However, the conclusions reached with such models are only locally necessary, in the sense that they may change if access to specific elements in episodic memory is altered.

At the next higher level of sophistication, individuals develop the ability to construct and use mental models that represent classes of events and relations derived from semantic memory (i.e. knowledge that is outside the realm of personal experience). As their standards of rigor increase further, individuals deem a conclusion valid only if their search for possible outcomes is systematic and exhaustive. They come to realize that, in some situations, possibilities may exist despite their personal inability to specify the exact nature of these possibilities. Finally, as they become acculturated into the ways of formal science, mathematics, or logic, individuals learn and attempt to make their reasoning conform to systematic principles governing validity. That is, they attempt to utilize formal deduction, logic, and axiomatic thinking.

## Conclusion

As students manipulate and reflect on their manipulations of a Shape Maker, they abstract certain actions that they are able to perform with it. They integrate these abstractions into a mental model of the Shape Maker that constitutes the students' construction of meaning for the Shape Maker. This mental model includes representations of the Shape Maker's visual characteristics and movement possibilities, which, because of the way the Shape Maker has been constructed, reflect the geometric properties of the class of shapes made by the Shape Maker. At first, these properties are incorporated into a mental model implicitly as the properties become embodied in the model's simulated behavior. Later, the properties become explicit as they are individually disembedded and abstracted from the simulated behavior, and as students develop, through social interaction, the terms and concepts used to describe them in traditional mathematical language. What results from the increasingly sophisticated mental models of Shape Makers is not only knowledge of geometric properties of shapes, but the ability to reason about shapes and classes of shapes in increasingly sophisticated ways.

## References

Battista, M. T. (in press). SHAPE MAKERS: Developing Geometric Reasoning with *The Geometer's Sketchpad*. Berkeley, CA: Key Curriculum Press.

Battista, M. T. (1994). On Greeno's environmental/model view of conceptual domains: A spatial/geometric perspective. *Journal for Research in Mathematics Education, 25*, 86-94.

Clements, D. H. & Battista, M. T. (1992). Geometry and spatial reasoning. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 420-464). New York: NCTM/Macmillan.

Greeno, J. G. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education, 22*, 170-218.

Johnson-Laird, P. N. (1983). *Mental models: Towards a cognitive science of language, inference, and consciousness.* Cambridge, MA: Harvard University Press.

Markovits, H. (1993). The development of conditional reasoning: A Piagetian reformulation of mental models theory. Merrill-Palmer Quarterly, 39(1), 131-158.

## Notes

Explicity knowledge may or may not be incorporated into one's mental models.

How is explicity knowledge incorporated into our mental models?

Conceptualizations include our explicity knowledge. Mental models somehow exist at deeper level and can be used in our conceptualizations.

Sometimes knowledge that we have can't be integrated into a proper mental model—3d for example.

# INTERACTIVE DIAGRAMS: A NEW LEARNING TOOL

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This research interviews four students working with an interactive diagram which relates the coefficients of the general form of the quadratic functions to a graph. The students' investigation of b and c is discussed.

Each time a new medium becomes available, new representational forms evolve. As we have worked for the past year on *Multimedia Precalculus* (Confrey and Maloney, in progress) we have made considerable use of a new kind of tool for communicating mathematical ideas, called the "interactive diagram" (an ID). An interactive diagram is a computer-based representation that permits a user to undertake investigations of a display on the screen by varying certain selected parameters. Unlike a static figure and picture, it's interactivity allows the user to try out variations and observe consequences. Unlike a tool like FP2 (Confrey, 1996), where the students' choices of actions are quite flexible and varied, in the ID, the possible actions are constrained to modifying a few parameters. Because of this constraint, we refer to an ID as a "closed tool".

Through our design process we have come to identify and design for three overlapping types of interactive diagrams: simulations, representations, and illustrations. In this paper we focus on representations that look more like traditional mathematical expressions but which permit rapid actions on them and machine responses. The one that we will discuss in this paper included a graph of a parabola and its equation in general form with adjustable coefficients. We choose to report on research on "representations", because although relatively traditional looking, they yield surprising inventions and insights.

## The Study

Four college students from Cornell University were interviewed for two hours over two days. The small sample consisted of two women and two men. The group included an Asian and an African-American woman, and a Puerto Rican and White male. All students were completing a precalculus course taught with computer-based labs and a focus on (a) contextual problems, (b) multi-representational software, and (c) transformations generally of the form $y = A *f( B(x-C)) + D$. Three IDS were discussed, but only one is presented herein.

## Methods

The students volunteered for the study over exam period and were paid for their participation. Students were asked to use the ID and were told that the diagrams were proto-types in the larger multimedia project and would benefit from their feedback. In each case they were given a mathematical task to do or explain something, so discussion of the IDs could accompany the process of solution. The research questions the team sought to answer were three: (1) how did the student approach the task? (2) how did the ID seem to influence the approach? and (3) how could the ID have assisted the student in his/her approach? The interviewer's primary role was to follow the approach of the student. The diagram of interest was of a graph of the function: $y = ax^2+bx+c$. The equation was written so that there were three boxes for $a$, $b$ and $c$ that could be varied. The initial conditions were $a=1$, $b=0$, $c=0$ and the ranges were $-4 \leq a \leq 4$ , $-8 \leq b \leq 8$, and $-8 \leq c \leq 8$. The fixed range on the graph displayed from -8 to 8 with units of 2. There was a reset button. In the quadratic there was also a trace feature called "erase" which when *not selected*, left an image of each of the previous parabolas that were graphed until reset was clicked. In each case, the student was asked to vary the coefficients using the box or slider and to predict and describe how the graph would change.

## Results on the General Form of the Parabola Coefficients

The majority of the interview was spent working with varying the coefficients in the general form of the quadratic $y = ax^2+bx+c$ . Because the students were far more accustomed to the form $y = A(x-h)^2+K$, (we will call this a transformational form) none of the students found the task easy; nor did any fully investigate all the interactions among $a$, $b$, $c$. All students eventually defined their task as relating these two forms.

Typically, a quick answer to this task by instructors is that a controls the shape of the curve, c predicts the y intercept and b is difficult to describe. However, from previous research by Borba (1995), we knew that by systematically varying $b$, a family of parabolas were produced whose vertices, if connected, create a parabola. Furthermore, from another study by Borba (1997) with graphing calculators, we knew if $b$ and $c$ were not equal to zero, then a would not only effect curve shape but the location of the vertex. This previous work alerted us to the fact that understanding changes in the coefficients was not a trivial task.

In our results, we will describe two approaches to discussing the changes in the coefficients and the location of the vertex. We will then offer a more elaborated analysis of the task and discuss the issues of how the ID was used and how it could have been used. This approach to analysis is within a heuristic proposed by Confrey (1994) called voice and perspective dialectic where one first attempts to authentically present the approach of the student as it evolves in the interview, "voice". And then one steps back and explicitly allows the experience of "voice" to lead one to explicit and deep reexamination of the content, "the perspective" part of the dialectic. It is a dialectic, because the interactions occur repeatedly and because the cycle of analysis is repeated multiple times.

As a result of our analysis, we have come to conjecture that in these few interviews, we saw evidence of three systematic ways to work in this closed tool environment: (1) conjecture and confirm/disconfirm (with a limited number of cases) (2) coordination of forms through algebraic bridging, and (3) dynamic visualization.

*Student Voice I*: Paula (Asian, female)

Paula's initial approach to the task was to explore $a$, $b$ and $c$ dynamically, changing the coefficients using the sliders and looking at what happened with the graph. She quickly was able to relate $a$ to the stretch factor and $c$ to both the $y$ intercept and the vertex. She begins to change $b$, and describe this movement in terms of slope. The interviewer was surprised by her use of slope and unsure whether her references would prove understandable.

When asked about what she meant by slope she said that she meant the change of $y$ over the change of $x$ [as a description of the change in the location of the vertex]. She showed surprise when she changed $b$ to $4$, because the vertex was at $(-2,-4)$ and not at $(-1,-4)$ as she expected. She then conjectured that the slope will be $(b/2)^2$. She was able to predict the behavior of the graph for some b values, including negative ones, but she kept saying that she was confused as to why the graph moves to the third quadrant when $b$ is positive and to the fourth quadrant when $b$ is negative. She suggested that it is because of the value of the slope. She said that if $b$ is positive, the slope should be in one direction, and if it is negative it should have opposite directions, and signaled the directions with her hands. She kept repeating that she was not used to the general formula, and that she would prefer to work with the transformational one. Earlier she suggested that she could work from the general form to the transformational form, but only pursued it at the suggestion of the interviewer. From completing the square algebraically, she concluded at the end that the vertex is $(-(b/2), -(b/2)^2)$. She was very excited about it. She then predicted the vertex for $a=2$, $b=-6$ and $c=3$. Her prediction differed from ID having forgotten to consider the impact of a on the vertex. She works algebraically again to correct her prediction. Near the end of the investigation, the interviewer had her reexamine her initial conjecture about slope. She had shown that when you change from general to transformational form, then $(h, k)=(-(b/2a, -(b/2a)^2)$[1]. Treating this as a slope, give you a slope of the line connecting $(0, 0)$ to $(h, k)$ as $(b/2a)$. In the case that $a=1$, then it is in fact a slope of $b/2$ as she predicted.

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[1] Her note show that she found the $k=-(b/2)^2/a$, but he makes a mistake and puts the a inside the parenthesis.

At the end of the interview she mentioned that she was not sure how much the ID helped her to understand the quadratic, but when the interviewer showed how you can change the graph dynamically, she seemed excited and said that the graph was going down like an inverted parabola form. When prompted to look to her vertex formula, she said it makes sense because there is a square and a negative sign which makes the parabola inverted. She ended her interview by mentioning that working on this was terrific.

*Student Voice 2*: Carl (Puerto Rican, male):

Carl also began by looking to the graphs dynamically. He soon changed to inputting the values to obtain just the graphs he wanted. He said that changing the a changes the width and making it negative causes a reflection. Next he tested $b$, but seemed surprised by what $b$ did. He too said he was used to the transformational formula $y=A(x-h)^2 + k$, where A gives you the stretch, $h$ and $k$ the coordinates of the vertex. Then he picked up the transformational form and expanded it. Because he wrote the transformational form as $y=-A\ (x-h)^2 + k$, he produced the equation, by hand, $y= -Ax^2 + 2Axh - Ah^2 + k$ . Because he could not immediately see how this helped to answer the question of what $a$, $b$ and $c$ do, he worked two examples: $y=2\ (x+3)^2 + 2$ and $y=2\ (x+1)^2 + 2$ and changed them into general form. He explored the graph trying to find relations between $a$, $b$ and the vertex. He showed confusion on the explorations and affirmed that he could predict everything for the transformational formula, but that he was not used to the general formula for graphing, but only for algebra problems. He turned to explore $c$ saying it tells you where the vertex is going to be in the $y$ axis [He seems to be thinking only of quadratics where $b=0$ but doesn't specify this.]. He conjectured that $b$ will give you where the vertex is going to be on the $x$ axis. He comes back to the formula trying to establish a relation between $b$ and $c$. He says he thinks $a$ is the same as $A$ because it does not change, just get distributed

He turned his attention to the relationship between $c$ and $k$. By examining his two examples, he saw that $k \neq c$. He looked at the process of squaring the $(x+2)$ and the $(x+1)$ terms and saw that they produced a constant term that was subsequently added to $k$. He conjectured therefore that the problem relates to $b$. he said that $k$ is doing something with $b$ to get the vertex. This led him to think, perhaps the relationship was that $k= b/c$ so he inputted $b= 6$ and $c = 3$ to test his conjecture $(a=1)$. To his surprise, the vertex of the parabola moved to $(-1, 5)$. He looked again and proposed an alternative conjecture, that if you add the coefficients of the vertex ($h$ and $k$ in more formal terms) then you get $c$. The idea that $c$ is the sum of the coefficients in the vertex surprised the interviewer, who initially found the idea of adding an ordered pair a very unlikely conjecture. Because of the negative sign, he adjusted his conjecture to take the absolute value of $h$ and add it to $k$. He tested the equation $y = x^2 + x + 4$ . The vertex, he said was approximately $(-.5, 3.5)$, although he was not certain of its accuracy. By this time, the interviewer has reexamined the term $(-Ah^2 + k)$ in the expansion of transformational form, realized that since $-A = 1$, then this simplifies to $h^2 + k$ and that if he squares rather than takes the absolute value, his conjecture will work. What surprised her is that thinking of this algebraic value of $c$ operationally as an additive action on the ordered pair of the vertex never occurred to her. She suggested that he reexamine his earlier formula for the transformational expansion , $y = -Ax^2 + 2Axh - Ah^2 + k$, and he now could interpret it. He concluded that $c = -Ah^2 + k$. and then he sees that if he rewrites it using $A$ and not $-A$, he has $y = Ax^2 - 2Axh + Ah^2 + k$, where $a = A$, $b = -2Ah$ *and* $c = Ah^2 + k$. He computes the vertex for $a=2$, $b= 6$ and $c=3$, but makes a computational error, which he corrects after he checked his computation with the graph produced on the ID.

## Perspective

Our understanding evolved by watching and analyzing the tasks. Both students found it necessary to link the general form and the transformational form algebraically in order to explain the impact of $a$, $b$, and $c$. Paula went from general form to transformational form and Carl did the opposite. Either of these algebraic manipulations produces all the information needed to link the coefficients $a$, $b$ and $c$ to the ordered pair for the vertex $(h, k)$ and the stretch factor $A$. Because of this, predicting the vertex from the coefficients in general form is "a solved problem", at this point.

However, we suggest in a dynamic environment, this is only the beginning of the solution to the question, "how does changing the coefficient have an impact on the parabola?" Our further question is "what can be gleaned from these algebraic equations in a dynamic environment?" To explore the diagram more fully, one can consider not only how to predict the vertex and shape of the parabola, but also what sequence of curves are produced when sequences of values are substituted for $a$, $b$ or $c$. We refer to this as "dynamic visualization" which means parameterizing the coefficients and examining the resulting families of curves. Because of the special qualities of zero, this investigation is often aided by setting one of the other two coefficients to zero to reduce the interactions among the variables.

Neither student undertook full dynamic visualization, because both treated their algebraic statements as formula and not functions. Their tendency was to just input convenient or "nice" values for $b$, but not to input a sequence of values or to use the increment capability of the slider to produce the family visually after doing the algebra. They also did not see it because few if any curricula teach students to consider how to envision implicit formula in terms of the characteristics of functions which relate different variables (direct, indirect, squaring, etc.).

A second issue besides parameterizing is a question of the order of the replacement of the coefficients. In a static medium all three changes are input and present simultaneously. However, in Paula's case, since she felt that she understood c as a vertical translation of $c$ on $y= ax^2 + bx$, she only needed to learn to predict the vertex in that simpler case $(c=0)$. We referred to this as *order of replacement* of the coefficients.

Furthermore, we are suggesting that the way the diagram is used varies across students. Some use it as simple conjecture and confirm/disconfirm tool. Others can become deeply involved in coordinating graphical and algebraic forms through a careful examination of the algebraic equivalencies as bridged formulaically. Finally, we propose a newer form of "dynamic visualization" by which the user parameterizes the coefficients in order to consider what family of functions evolves. When multiple coefficients are involved, there is a further issue of or ordering one's replacement of coefficients.

There are design modifications which must be undertaken to increase the potential of the diagram. These include simple revisions such as adding a register of the vertex and a point one unit left and right of it. It includes more difficult design challenges such as building a "scriptable slider" that allows the user to create the parametric sequence to fit the particular situation. And, it includes a question of how to let the user create and specify a generalizable method that includes the order of replacement.

## References

Borba, M. (1997). Graphing calculators, functions and reorganization of the classroom. In M. Borba, T. A. Souza, B. Hudson, & J. Fey (Eds.), *Proceedings of the of the Working Group: The role of technology in the mathematics classroom*. Rio Claro, Sao Paulo: UNESP, State University of Sao Paulo, pp. 53-60.

Borba, M. (1995). Students' understanding of transformations of functions using multi-representational software. (Doctoral dissertation, Cornell University, Ithaca, NY, 1993). Lisbon, Portugal: Association of Mathematics Teachers of Portugal.

Confrey, J. (1991-96). Function Probe© [Computer Program]. Dept. of Education, Cornell University Research Foundation.

Confrey, J. (1994). Voice and perspective: Hearing epistemological innovation in students' words. In N. Bednarz, M. Larochelle, and J. Desautels (Eds.), *Revue des Sciences de l'education*, Special Issue: Constructivism in Education, 20 (1), pp. 115-133.

# CONJECTURING AND REPRESENTATIONAL STYLE IN CAS-ASSISTED MATHEMATICAL PROBLEM SOLVING

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**This study examined the ways in which college students with substantial mathematics backgrounds and access to calculators with computer algebra systems (TI-92s) generate and verify conjectures in their non-routine problem solving. Eight college students who had completed at least 15 credits of college mathematics were observed as they solved non-routine problems in the context of a series of content-based interviews. Students displayed representational styles in the ways they generated conjectures and in the ways they verified them. Most students were likely to seek a verification for a conjecture in the same representation that prompted the conjecture. In addition, students' relationship to the technology and the extent to which they were prone to errors influenced the ways in which they made and tested conjectures.**

## Background

Conjectures, for the purpose of this study, are mathematical generalizations "that may be true or false; at the time of consideration, the conjecturer does not know for sure whether it (the conjecture) is true or false, but thinks that it is true" (Chazan & Houde, 1989, p. 3). Generating and testing conjectures are processes that are essential in doing mathematics (Schoenfeld, 1994). Multirepresentational hand-held technology presents promising opportunities for students to generate data and produce mathematical conjectures about relationships in that data. Research on technological approaches to the teaching and learning of mathematics (Chazan, 1993; Yerushalmy, 1993; Yerushalmy et al., 1993) have begun to reveal potential barriers to engaging students in such authentic mathematical activity, yet most of this research has been conducted with high school students who have little exposure to mathematics. This study will examine the ways in which college students who have substantial mathematics backgrounds and access to multirepresentational technology generate mathematical conjectures and test them.

## Sample and Instructional Setting

Subjects for the study were drawn from the 17 prospective secondary mathematics teachers who enrolled in the only section of a mathematics education course focusing on technology and the learning and teaching of mathematics (TLTM) (all volunteered to participate in the study). TLTM had a prerequisite of 10 credits in post-calculus college mathematics. Eight students were selected for this study, on the basis of their performance on a non-routine problem-solving task, to represent a range of tendencies to explore and investigate mathematics.

The TLTM course met for 15 three-hour weekly sessions during the semester. Each student was provided with a TI-92 calculator[1] for use both in and out of class. One primary aim of the course was to provide the students the opportunity to deepen their understandings of concepts of school mathematics in a technological environment. Students explored mathematics through open-ended assignments in which they were expected to justify their claims and to reflect on their thinking. One of the authors of this paper was the course instructor, and a second had served as course instructor during the pilot semester. These mathematics education faculty, along with five mathematics education and science education doctoral students and one mathematics professor constituted the research team[2].

## Data Sources, Collection, and Analysis

The major source of data for the study reported here is a series of four task-based interviews conducted by members of the research team with the targeted students. The task-based interview schedules, each of which included two non-routine problems[3], were constructed by the research team. The interview tasks required mathematical explorations and were designed to be amenable to solution through use of the TI-92. Tasks required exploring families of functions, exploring geometrical relationships, and creating and exploring mathematical models. The project team piloted all interview tasks and data collection techniques with the TLTM class offered in the previous semester. The interviews were audiotaped and videotaped, and verbatim transcripts were produced, proofed, and annotated with calculator screen dumps and descriptions of actions. The project staff read through the transcript of each interview task for each subject while viewing the videotape. Preliminary passes identified individual student conjectures and characterized students' generation and verification[4] of conjectures. Subsequent comparative passes allowed the emergence of themes relating students' conjectures to use of representations and technology (Glaser & Strauss, 1967).

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[1] Among the utilities on the Texas Instruments TI-92 calculator are a computer algebra system and a dynamic geometry tool. This material is based on work supported by the National Science Foundation under Grant No. GER 94-54048, by an equipment loan from Texas Instruments, and by Department of Curriculum and Instruction of The Pennsylvania State University. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the sponsors.

[2] In addition to the authors, Robert Kuech, James Laughner, and Ron Wenger served on the research team.

[3] The problem statements appear in a previous PME Proceedings, (Blume, Heid, Iseri, Kuech, Laughner, Marshall, Wenger, 1996).

## Results

Subjects displayed the ability to form conjectures based on a wide array of sources (numerical, graphical, symbolic); however, dominant representational styles characterized the conjecturing and verifying of each subject. Sample representational styles are discussed in the following section. In addition to representational style, students' strategies for using technology and their tendencies to make errors seemed to influence the ways they made and tested conjectures.

### Representational Style

Amanda frequently shared her view that she needed to search for a symbolic representation, claiming that it was the best representation. Arriving at the symbolic representation, however, did not seem to convince Amanda, and she did not seem to reason from the symbolic representations she generated. Instead, pictures and graphs provided Amanda the most powerful representation with which to reason. She used the graphing capabilities of the calculator to help her make conjectures and to guide her in reasoning symbolically about them. Frederick's preferred representation was symbolic. He generally started with the symbolic, made conjectures based on the symbolic, and often returned to the symbolic to verify his conjectures. Nevertheless, Frederick did not confine his verifications solely to the symbolic. Frederick verified conjectures using a variety of representations and drawing on his well-connected mathematical knowledge. Hillary's forte was numerical data. Numerical data allowed Hillary to form conjectures across most of the interview tasks. For example, she generated two rules from a patterned counting strategy, one symbolic and the other iterative, and she verified both of these rules numerically. Hillary's strategy generally consisted of taking small steps in her conjectures and verifying each of these steps. The deliberation with which she undertook each step may have been a reflection of her hesitation to make large conjectures without being confident of their truth or feasibility. Agnes' conjectures generally arose out of a search for patterns in data. She reasoned fairly well about the patterns she observed, connecting her conjectures to mathematical concepts like rate of change. Although usually this data was numeric and displayed in a table, her search for patterns in data extended to graphic, numeric, and pictorial data.

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[4] For the purposes of this study, the researchers interpreted "verifications" to mean processes students used to satisfy themselves concerning the validity of their conjectures, rather than formal mathematical justifications.

In a surprisingly large number of cases, students verified a conjecture using the same representation they used to generate it. Agnes' conjectures tended to be based more upon patterns, no matter what the representation, and her verification of these conjectures was usually based in the same representation in which the pattern was first observed. Hillary's conjectures, almost always arising from the numeric were also almost always verified by the numeric. Debbie's conjectures were most often numerically or graphically based and usually, but not always, verified within the same representation from which they arose.

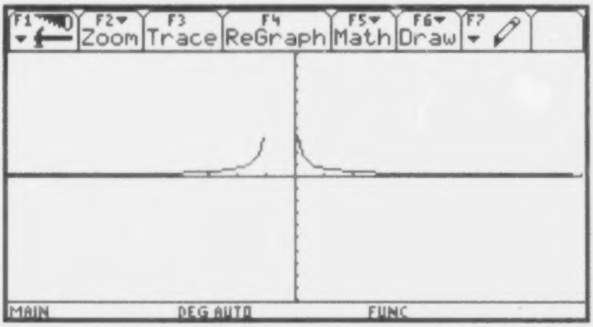
**Role of Technology**

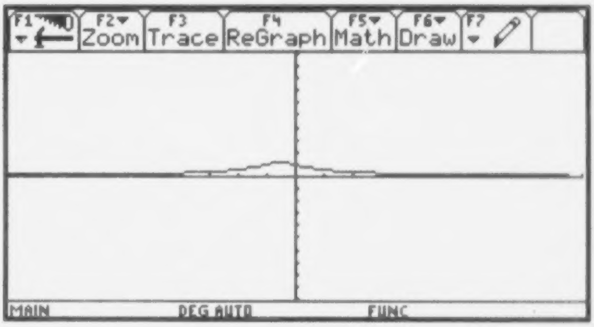
The TI-92 technology with its CAS and dynamic geometry capabilities played an important role in the ways in which students generated and attempted to verify conjectures. Perhaps because of its extensive and flexible capabilities, the technology provided Frederick with an arena in which to use his extensive repertoire of mathematical connections to verify his conjectures. Perhaps because of the calculator's ability to generate quickly a large number of examples, Debbie, Hillary and Agnes used it in their conjecturing to generate examples about which they reasoned as well as to facilitate their verification processes. Debbie and Agnes appeared to be more likely to use the calculator to verify a conjecture if they had utilized it generating the examples upon which the conjecture was based. Perhaps because of the speed with which the calculator produced results, Tim used the calculator handily in his routine of generating an example, making a conjecture, and testing to see if his conjecture was true, and then moving on to the next example and the next conjecture. The calculator was not always a help, however, and students were sometimes hindered in their conjecturing activities because of their knowledge or interpretation of calculator results or capacities. Throughout the interviews, for example, Jay displayed a penchant for working with the calculator, but his unfamiliarity with the meaning of calculator results disrupted his conjecturing on some occasions. Similarly, Emily's acceptance of calculator measurements without attention to the consequences of roundoff as well as her limited view of a graph resulting from overly restricted window settings led her misguidedly to reject one of her conjectures.

Knowledge of the meaning of calculator results affected some students' work with conjectures. During his work on the Polygons Task, Jay conjectured that in order for the perimeter of an inscribed polygon to be greater than 90% of the circumference of the circle, the polygon would have to have more than three sides. To check his conjecture, Jay constructed an inscribed pentagon, measured its perimeter, and noted that the perimeter was larger than 90% of the circumference of the circle in which it was inscribed. He outlined the rest of his verification as having to check the case of four sides, and three if four was still larger. When he checked the case of the inscribed square, the perimeter of the square displayed on the calculator screen had the same value as 90% of the displayed circle circumference (both were 9.95 cm). The machine values were rounded, however, and the square's actual perimeter exceeds 90% of the circumference. Jay did not interpret the values he saw as approximate, and he chose the regular pentagon as the first case that exceeded 90% of the circle's circumference. This error could have been resolved by displaying more precision in the measurements or by testing 4> 1.8π or a number of other ways including graphing. Jay seemed very confident in the accuracy of the results the calculator provided and did not seem to have the slightest question about his answer.

Frederick used technology in his conjecturing process in a variety of ways. In considering a family of functions task, Frederick had focused on determining the asymptotes of the family, $f(x) = \sqrt{\dfrac{1}{ax^2 + bx + c}}$

He conjectured that the graph of *f* would have asymptotes when the denominator was zero, and he proceeded to verify his conjecture through by-hand and calculator-generated symbolic manipulation. Later in the interview, he substituted in values, a = 1, b = 1, and c = 0, and conjectured that changing the *c* value would cause a shift in the graph. In the standard viewing window he saw the graph of y = (shown on the left in the following figure) and conjectured that the graph does not stop, but rather that there were vertical asymptotes at x = 0 and x = -1. He verified this conjecture several ways, by zooming in, tracing along the graph, and substituting in the values into his symbolic forms that he had derived for the asymptotes. He substituted in a = 1, b = 1, and c = 1 and then viewed the graph shown on the right in the following figure:

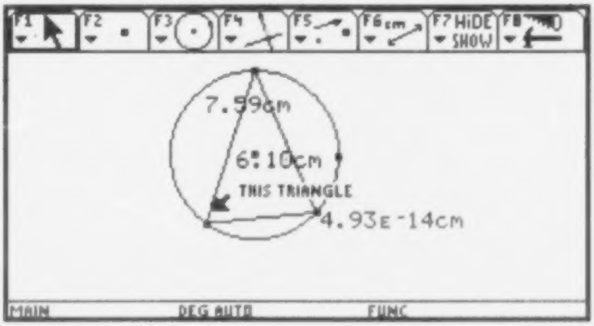


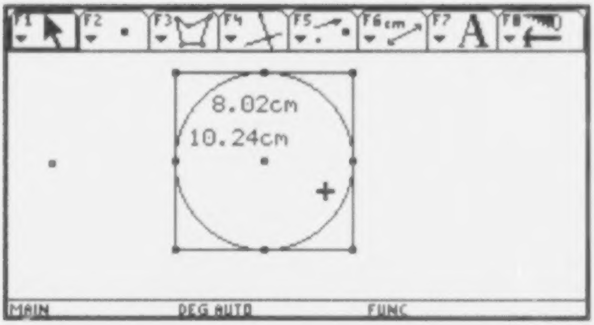


He had expected to see an upward shift in the graph, and he seemed surprised at the resulting graph as he remarked: "Oh, okay! c didn't, c had a different effect." To determine what effect c = 1 had, he reasoned symbolically. He concluded that there would not be any real asymptotes in the graph: "Since these are imaginary numbers, they don't exists on this graph, meaning that the values that would make this equal to zero don't exist . . . so we're not going to get any asymptotes."

**Role of Errors**

Because of her focus on patterns, Agnes' frequent errors in arithmetic, counting, and drawing seemed to have a significant effect on what she conjectured since these mistakes caused disruptions in her patterns. It is possible that Agnes' errors and her response to them played a role in her ownership of her conjectures and her readiness to abandon her conjectures. Anomalies in the patterns she saw did not seem to surprise her, possibly because of the frequency with which she encountered such errors. This readiness to abandon conjectures arose in Agnes' work even when her errors were conceptual in nature. It is interesting to compare Agnes' reactions with those of Debbie when both conjectured, because of a conceptual error, that the circumscribed square would more closely approximate the circumference of a circle than would an inscribed triangle since a square has more sides than a triangle. Debbie was satisfied enough with her conceptual argument that she did not attempt any verification of her conjecture. Agnes, on the other hand, rejected her conjecture on the basis of one poorly structured example. She had constructed the square and a non-equilateral triangle using circles of different sizes as shown in the following figure.


F1
F2
F3
F4
F5
F6
F7 HIDE SHOW
F8
7.59cm
6.10cm
THIS TRIANGLE
4.93E-14cm
MAIN
DEG AUTO
FUNC


F1
F2
F3
F4
F5
F6
F7
F8
8.02cm
10.24cm
MAIN
DEG AUTO
FUNC

C = 7.59 $P_D$ = 6.16 C = 8.02 $P_s$ = 10.24

## Conclusion

As the subjects in this study made and tested conjectures within the context of their mathematical investigations, two features of their problem solving style seemed to influence their thinking: their use of multiple representations and the ways they used technology to generate and test their conjectures. For many of the subjects, a marked preference for a particular representation seemed to dominate their generation of data from which to conjecture. Some of these subjects also seemed to rely on their preferred representation as the initial arbiter for the truth of their conjectures. This study begins an investigation of the ways in which students who have access to multirepresentational technology generate and test mathematical conjectures.

## References

Blume, G. W., Heid, M. K., Iseri, L., Kuech, R., Laughner, J., Marshall, J., & Wenger, R. (1996). A computer algebra system (CAS) learning study: The impact of access to a CAS calculator on the nature of mathematical exploration by prospective mathematics teachers. In E. Jakubowski, D. Watkins, & H. Biske (Eds.), Proceedings of the *Eighteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 541-546). Columbus, OH: ERIC.

Chazan, D. (1993). High school geometry students; justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics, 24*(4), 359-387.

Chazan, D. & Houde, R. (1989). *How to use conjecturing and microcomputers to teach geometry.* Reston, VA: National Council of Teachers of Mathematics.

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research.* New York: Aldine Publishing Company.

Schoenfeld, A. H. (1994). Reflections on doing and teaching mathematics. In A. H. Schoenfeld (Ed.), *Mathematical thinking and problem solving* (pp. 53-70). Hillsdale, NJ: Lawrence Erlbaum Associates.

Yerushalmy, M. (1993). Generalization in geometry. In J. L. Schwartz, M. Yerushalmy, & B. Wilson (Eds.), *The Geometric Supposer: What is it a case of?* (pp. 57-84). Hillsdale, NJ: Lawrence Erlbaum Associates.

Yerushalmy, M., Chazan, D., & Gordon, M. (1993). Posing problems: One aspect of bringing inquiry into classrooms. In J. L. Schwartz, M. Yerushalmy, & B. Wilson (Eds.), *The Geometric Supposer: What is it a case of?* (pp. 117-142). Hillsdale, NJ: Lawrence Erlbaum Associates.

# ROLES OF SYMBOLIC REPRESENTATION IN CAS-ASSISTED MATHEMATICAL PROBLEM SOLVING

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This study examined the ways in which college students with substantial mathematics backgrounds and access to calculators with computer algebra systems (TI-92s) use symbolic representations in their non-routine problem solving. Eight college students who had completed at least 15 credits of college mathematics were observed as they solved non-routine problems in the context of a series of content-based interviews. Students seemed to generate symbolic representations for different purposes. These purposes included having a symbolic representation to manipulate, serving as a place holder for an idea to which to return, and capturing their personal understanding of a concept. Although most of the subjects regularly adopted the goal of generating a symbolic representation, only one of the eight target students regularly reasoned from the representations he had generated.

## Background

With ready availability of multi-representational technology has come an increased interest in the ways in which students use multiple representations of functions to reason about mathematics. Although symbolic representations have long played a central role in students' mathematical development, most of the work to date on multiple representations of functions has centered on work with graphical or numerical representations (Dugdale, 1993; Goldenberg, 1995; Janvier, 1987; Kaput, 1986; Kaput, 1994; Nemirovsky, 1994; Perkins & Unger, 1994; Pimm, 1995). The advent of classroom-friendly computer algebra systems (CASs), however, generates a new interest in students' use of symbolic representations and how that use might influence their mathematical thinking.

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[1] This material is based on work supported by the National Science Foundation under Grant No. GER 94-54048, by an equipment loan from Texas Instruments, and by Department of Curriculum and Instruction of The Pennsylvania State University. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the sponsors.

Researchers are currently grappling with the issues of the nature of symbolic reasoning and symbol sense and the roles they will play in technology-intensive mathematics classrooms (Dreyfus & Halevi, 1990/91; Fey, 1990; Kaput, 1987; Keller, 1993/1994; Nemirovsky, 1994; Onslow, 1991; Pimm, 1995; Yerushalmy, 1992; Yerushalmy, 1993). A small but growing number of theoretical perspectives have been offered and need to be corroborated through observations of students using symbolic representations in their mathematical problem solving. This study[1] examines ways in which students with significant college mathematics backgrounds, and equipped with hand-held symbolic manipulators, generate and use symbolic representations in solving non-routine mathematical problems.

## Sample and Instructional Setting

Subjects for the study were drawn from the 17 prospective secondary mathematics teachers who enrolled in the only section of a mathematics education course focusing on technology and the learning and teaching of mathematics (TLTM). Each of these students was enrolled in a teacher preparation program and each had completed at least 15 mathematics credits at the Calculus level and beyond. All seventeen students volunteered to be observed in their in-class work and to participate in task-based interviews, and eight students were selected for observation and task-based interviews. These eight were selected, on the basis of their performance on a non-routine problem solving task completed as an assignment during the first week of class, to represent a range of tendencies to explore and investigate mathematics. The names used in this report are not the students' actual names but we have retained the gender of the subjects in the names we have chosen.

The TLTM course met for 15 three-hour weekly sessions during the semester. Each student was provided with a TI-92 calculator for use both in and out of class. One primary aim of the course was to provide the students the opportunity to deepen their understandings of concepts of school mathematics in a technological environment. Students explored mathematics through open-ended assignments in which they were encouraged to justify their claims using numeric, symbolic, and graphical representations and to reflect on their thinking. One of the authors of this proposal was the course instructor, and a second had served as course instructor during the pilot semester. These mathematics education faculty, along with six mathematics education and science education doctoral students and one mathematics professor constituted the research team[2].

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[2] In addition to the authors, Robert Kuech, James Laughner, and Ron Wenger served on the research team.

## Data Sources and Collection

The major source of data for the study reported here is a series of task-based interviews conducted with the targeted students. The interviews were audiotaped and videotaped. Transcripts were produced, with a tri-level procedure for checking the accuracy of the transcripts, by the mathematics education research staff who had conducted the interviews. First, an initial transcript of verbal interchange was prepared from the audiotape. Second, the initial transcript was proofread and annotated by comparing the transcript with the videotape and annotating it with calculator screen dumps and descriptions of actions. Finally, as a subgroup of the research team viewed the videotape while reading and analyzing the transcripts, they made additional corrections to the transcripts.

The task-based interview schedules, each of which included two non-routine problems, were constructed by the project team. The interview tasks required mathematical explorations and were designed to be amenable to solution through use of the TI-92. Tasks required exploring families of functions, exploring geometrical relationships, and creating and exploring mathematical models. The project team piloted all interview tasks and data collection techniques in the TLTM class offered in the semester previous to that of the study.

## Data Analysis and Results

The project staff read through the transcript of each interview task for each subject while viewing the videotape. Preliminary passes through the individual transcripts focused on uses of symbolic, numeric, and graphic representations and generated a framework which reflected characteristics of the students' use of various representations. Subsequent comparative passes focused on themes that cut across different representations and allowed those themes to emerge (Glaser & Strauss, 1967) . Many of the themes which emerged featured the role of symbolic representations and, in particular, the processes of generating and reasoning from symbolic representations.

### Generating Symbolic Representations

Students seemed to generate symbolic representations for different purposes. These purposes included having a symbolic representation to manipulate, serving as a place holder for an idea to which to return, and to capturing students' personal understandings of a concept.

One student regularly generated symbolic representations so that he could perform symbolic manipulations on them. Frederick expressed the opinion that symbolic representations were easiest, he generated them as his representation of first resort, and once he produced a symbolic representation he tended to focus his activities on performing various manipulations on that representation. This tendency to create and manipulate symbols frequently, but not always, allowed Frederick to gain insight into the mathematical relationships at hand.

Other students seemed to use symbolic representations as place holders to which they could later return and which they would refine in successive returns. These representations seemed not to be intended to represent the actual mathematical ideas the students were entertaining. Instead, these representations apparently reminded the students that, although the students had not yet represented the exact relationships, they knew the magnitude and position of what they intended and would later revisit to refine the representations. Early in the Square Wrapping Function Task, for example, both Hillary and Amanda created piecewise definitions for the targeted distance (See Figure 1 for Amanda's first version of a piecewise-defined function for the Square Wrapping Function Task.) which they said were not the actual function rules. Their definitions were not accurate representations of the actual distances but rather expressions that held the place of representations they would seek to develop later in the interview.

Sometimes, students had a clearly understood idea for which they did not have a readily accessible symbolic representation. In these cases, students created personal non-standard representations whose purpose seemed to be to hold onto the features of the concepts the students were trying to depict. These representations were not place holders in that the student intended them to represent their current understandings and did not intend to return later to revise or refine them. Hillary's personal graphical representation (See Figure 2.) was intended to show that as the number of sides on an inscribed regular polygon increases, the perimeter of the polygon gets increasingly closer to the circumference of the circle.

**Reasoning From and About Symbolic Representations**

Frequently, students created symbolic representations as if the creation of the representation was their final goal. Only one of the eight students (Frederick) regularly reasoned from symbolic

representations, or used the symbolic to inform himself about other representations. For example, as Frederick explored a family of functions and noticed a vertical strip on the graph which contained no function values, he verified from his symbolic representation that the function was undefined in that region. More often students would work from another representation (usually, numeric or graphic) to verify pieces of information they were gleaning from the symbolic. Amanda, for example, used the numeric to get a handle on how the symbolic was working in her exploration of the family,

$$f(x) = \sqrt{\frac{1}{ax^2 + bx + c}}$$

She started with the case for which $a = b = c = 1$, and examined

A function f is defined which assigns to each angle u the shortest distance from the line L to the corresponding point P. Describe the behavior of the function f in as many ways as you can.

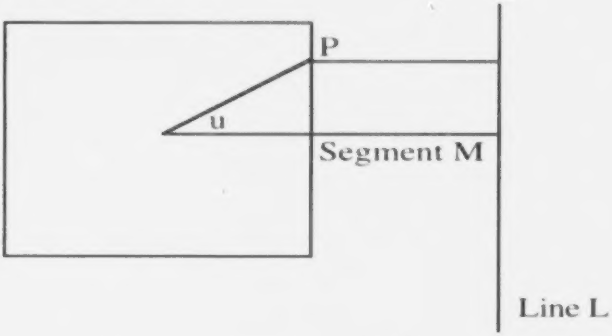

**Figure 1a** *Statement of interview task: Square Wrapping Function Task.*

$$f\left(\begin{array}{c|c} u \le 45^\circ & \overline{PQ}\ \text{Distance} \\ 315^\circ \le u^\circ & \text{fixed} \end{array}\right)$$

$$f\left(\begin{array}{c|c} 225^\circ \ge u \ge 135^\circ & \begin{array}{c} \overline{PQ} \\ \text{fixed Distance} \\ \overline{AB} + \overline{BQ'} \end{array} \end{array}\right)$$

$$f\left(\begin{array}{c|c} 135^\circ \ge u \ge 45^\circ & \overline{PQ}\ \text{Distance} \end{array}\right.$$

$$f\left(\begin{array}{c|c} 315^\circ \ge u \ge 225^\circ & \overline{PQ}\ \text{Distance} \end{array}\right.$$

**Figure 1b** *Amanda's work.*

**Figure 1** *Amanda's first pass at symbolizing the function for the Square Wrapping Function Task.*

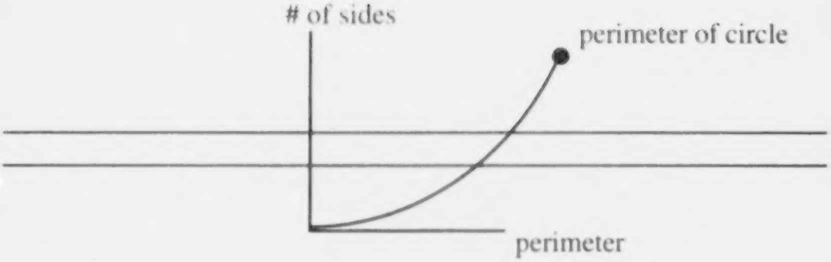

**Figure 2** *Hillary's personal graphical representation.*

$(-2)^2 + (-2) + 1$, $(-\frac{1}{2})^2 + (-\frac{1}{2}) + 1$, and $(-\frac{7}{8})^2 + (-\frac{7}{8}) + 1$, to give her a feel for how the $x^2$ term and the x term interacted. Only then did she move on to make some statements about the interaction among terms. When Jay began to reason from symbolic forms during the Circles Task[3], he rejected his symbolic results in favor of contradictory numerical or graphical results without reconciling the differences suggested by the various representations. On the Circles Task, Hillary zoomed in several times to determine whether the function rules she had produced to fit sets of data ($f(x) = 3x - 3$ and $f(x) = 3x - 11.5$) were parallel. There was a consistent pattern in the interview data, that although students saw the symbolic as a goal, they did not reason from it.

## Conclusion

As we studied the use of symbolic representations by students who have substantial collegiate mathematics backgrounds, we identified ways in which symbols commonly serve students in their problem solving. First, our students readily adopted the production of a symbolic rule as a goal in their problem solving; surprisingly, however, they did little reasoning from those rules once they had produced them. Second, just as earlier research suggested that paper-and-pencil arithmetic algorithms may alleviate the need to hold a large amount of information in memory, students' use of place holders may also serve to reduce the cognitive load in more complicated mathematical work. Third, in spite of a mathematical lifetime of exposure to standard mathematical notations and representations, students used personally designed non-standard representations to hold the mathematical ideas they were developing. As a result of these observations, we came to view the use of symbols by these students as an important part of their problem-solving process rather than simply a translation and manipulation procedure. This study deepens our understanding of the use of symbolic representations, an understanding that underpins much of mathematical thinking at the secondary levels and beyond.

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[3] The Circles Task along with the other tasks appear in a previous PME Proceedings, (Blume, Heid, Iseri, Kuech, Laughner, Marshall et al., 1996). It was created by Ron Wenger, University of Delaware.

## References

Blume, G. W., Heid, M. K., Iseri, L., Kuech, R., Laughner, J., Marshall, J., & Wenger, R. (1996). A computer algebra system (CAS) learning study: The impact of access to a CAS calculator on the nature of mathematical exploration by prospective mathematics teachers. In E. Jakubowski, D. Watkins, & H. Biske (Eds.), *Proceedings of the Eighteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 541-546). Columbus, OH: ERIC.

Derives, T., & Halevi, T. (1990/91). QuadFun—A case study of pupil computer interaction. *Journal of Computers in Mathematics and Science Teaching, 10*(2), 43-48.

Dugdale, S. (1993). Functions and graphs—Perspectives on student thinking. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp. 101-130). Hillsdale, NJ: Lawrence Erlbaum Associates.

Fey, J. T. (1990). Quantity. In L. A. Steen (Ed.), *On the shoulders of giants: New approaches to numeracy* (pp. 61-94). Washington, DC: National Academy Press.

Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research.* New York: Aldine Publishing Company.

Goldenberg, E. P. (1995). Multiple representations: A vehicle for understanding understanding. In D. N. Perkins, J. L. Schwartz, M. M. West, & M. S. Wiske (Eds.), *Software goes to school: Teaching for understanding with new technologies.* New York: Oxford University Press.

Janvier, C. (Ed.). (1987). *Problems of representation in the teaching and learning of mathematics.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Kaput, J. J. (1986). Information technology and mathematics: Opening new representational windows. *Journal of Mathematical Behavior, 5*, 187-207.

Kaput, J. J. (1987). Toward a theory of symbol use in mathematics. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of* *mathematics* (pp. 159-195). Hillsdale, NJ: Lawrence Erlbaum Associates.

Kaput, J. J. (1994). The representational roles of technology in connecting mathematics with authentic experience. In R. Biehler, R. W. Scholz, R. Strasser, & B. Winkelmann (Eds.), *Didactics of mathematics as a scientific discipline*. Dordrecht: Kluwer Academic Publishers.

Keller, B. A. (1993/1994). *Symbol sense and its development in two computer algebra system environments*. Unpublished doctoral dissertation, Western Michigan University.

Nemirovsky, R. (1994). On ways of symbolizing: The case of Laura and the velocity sign. *Journal of Mathematical Behavior, 13*, 389-422.

Perkins, D. N., & Unger, C. (1994). A new look in representations for mathematics and science learning. *Instructional Science*, *22*, 1-37.

Pimm, D. (1995). *Symbols and meanings in school mathematics*. London: Routledge.

# THE ROLE OF TECHNOLOGY IN LESSON PLANNING: THE CASE OF PRESERVICE TEACHERS

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This presentation describes preservice teachers' ideas of how technology can be used to help high school students develop an understanding of the mathematical concept under consideration. The subjects in the study were the five preservice teachers enrolled in the methods course, Teaching Mathematics with Technology. The analysis of the lesson plans developed by the subject revealed the differences in preservice teachers' ideas of the role of technology. The NCTM *Standards* (1989) maintain that the appropriate integration of technology will "transform the mathematics classroom into a laboratory . . . where students use technology to investigate, conjecture, and verify their findings . . . [and] the teacher encourages experimentation" (p. 128). In this context, teachers often become students as they struggle with understanding technology and its pedagogical implications (Norman, 1993). Vygotsky's assumption of an *active individual* and an *active environment* (Vygotsky, 1986) served as the basis for our hypothesis that teachers will learn about pedagogical implications of technology through their active involvement in lesson planning.

The results of the qualitative analysis show a continuum of meaning among preservice teachers with respect to how they perceive the role of technology in their classrooms. Two distinctive categories emerged: (i) technology that enhances the presentation of information; (ii) investigations for students to explore mathematics using technology. Further analysis revealed that the investigations ranged from simple to complex. Additional results and their implication to practice will be discussed at the conference.

## References

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Norman, F. A. (1993). Integrating research on teachers' knowledge of functions and their graphs. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions.* Hillsdale, NJ: Lawrence Erlbaum.

Vygotsky, L. S. (1986). *Thought and language* (rev. ed.). (A. Kozulin, Trans.). Cambridge, MA: M.I.T. Press.

# TEACHER AS BRICOLEUR: USING COMPUTING TECHNOLOGIES FOR TEACHING MATHEMATICS

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The objective of this study was to understand more about mathematics teachers' instructional decisions and practices involving computing technologies (CTs). Six mathematics teachers from two high schools were studied using survey, interviews, and observations. After observations of lessons on topics where teachers judged CTs to be most effective, they were interviewed. This interview-observation-interview cycle was repeated several times. Theoretical sampling of twenty additional teachers was used to further explore findings about the CT effective topics, assessment, and the role of collegiality in a teacher's CT use.

A primary finding was the usefulness of the metaphor of teacher as bricoleur (Lèvi-Strauss, 1966; Wheeler, 1980; Huberman, 1993) for understanding how mathematics teachers teach using CTs. Bricoleur is a French word that roughly translates to "craftsman" (Huberman, 1993). This metaphor was used to explain how these teachers viewed their "universe of instruments" and "rules of the game" (Lèvi-Strauss, 1966, p. 17) for teaching using CTs. Other findings of interest were those topics these teachers judged CTs to be most effective and the reasons teachers selected those topics as effective sites for CT use. These teachers judged CT use to be effective because CTs provided more accurate ways of doing mathematics, access to more mathematics, and an alternate method for teaching a topic regarded as difficult.

## References

Huberman, M. (1993). The model of the independent artisan in teachers' professional relations. In J. W. Little & M. W. McLaughlin (Eds.), *Teachers' work: Individuals, colleagues, and contexts* (pp. 11-51). New York: Teachers College Press.

Lèvi-Strauss, C. (1966). *The savage mind.* (Weidenfeld and Nicholson, Trans.). Chicago: University of Chicago Press. (Original work published 1962).

Wheeler, D. (1980, June). *A mathematics educator looks at mathematical abilities.* Paper presented at a Conference on Mathematical Abilities, University of Georgia, Athens.

# GRAPHING CALCULATORS IN THE MATHEMATICS CLASSROOM

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The teaching of mathematics has undergone many changes in the last three decades. Technologies are being integrated into mathematics classrooms nationwide with the full support of the National Council of Teachers of Mathematics (NCTM, 1989). This study explored factors that contributed to calculator proficiency to determine if there was a gender difference in calculator use.

This field study took place on a large high school campus that regularly employed graphing calculators. Observations and interviews were done in three classrooms, two advanced and one remedial. The data revealed: (1) Teachers report male students used the graphing calculator more, and were more familiar with it. "The boys seem to do more playing with the calculator." (2) The researcher ascertained from student interviews that the male students had more experience with this type of technology than the female students. (3) Video taped observations showed that when the calculator was shared between a male and female student, the male student *always* took control of the instrument. This was aided by the female student's passive non-engagement with the calculator.

The data reveal a gender difference in calculator use. According to Malcolm (1993) experience with an instrument will ameliorate anxiety. If the curriculum for advanced mathematics classes requires the mastery of certain technologies, then we must understand the ways in which our female students approach the tools involved and be willing to proactively adjust our pedagogy to induce their engagement.

## References

Malcolm, S. M. (1993). *An assessment of programs that facilitate increased access and achievement of females and minorities in K-12 mathematics and science education.* Washington, DC: American Association for the Advancement of Science. Office of Opportunities in Science.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

# CABRI-GEOMETRY: A CATALOG OF PROBLEMS THAT IT SOLVES AND QUESTIONS WHICH IT PROPOSES

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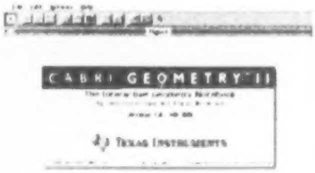
In the traditional curriculum, the interaction between the student and the geometric objects takes place via "photographic images" developed step-by-step in static form on the blackboard, without chance to change the position or the relative size between the elements the same construction. A dynamic geometry environment like CABRI-Geometry allows one to move and change the objects (for example: the center of a circumference can be moved within the environment; the focus and/or the directrix of a parabolic curve can be moved, etc.).



Traditionally, the teacher shows one figure and asks the pupils for calculations and/or the reasons about this image. If the teacher intends to give more graphic information, it must be obtained by additional constructions. This process is a laborious thing and it does not allow students to see the gradual gradual changes involved. In Dynamic Geometry setting, the graphic information is very important: when we obtain the first figure, we can then change the independent objects, keeping track of the links and invariants as students explore.

The group of problems to be shown has a status in the curriculum. Examples of the problems are:

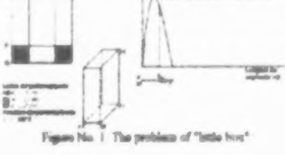


Figure No. 1 The problem of "little box"

| | |
|---|---|
| Optimization: | The problem of the "little box" (See Figure 1.); the rectangle of maximum area and fixed perimeter; the maximum angle to visualize a segment (over an inclined line); the cone with maximum volume starting from a circle; etc. |
| Functions: | The function of area of a sector between a circum- |

ference and a line; the exponential function drawn with traces on the axis; the construction of a general function with the commands calculate and tabulate; the tangent line in the optimization of one and two variables; etc.

Constructions: Mechanical curves, simulation problems in 3D, etc.

In an environment generated with CABRI-Geometry, questions that students pose and discuss include: what are dependent and independent objects?; If I move this element, what does it change in the figure?; Why does an element keep a specific link with another?

The creation of a catalog of environments for mathematical exploration could be an important goal for the teacher of mathematics. The catalog could help students produce observations, conjectures, proofs, new constructions, and propositions. Such an effort demands educational experimentation in the curriculum.

# GENERATING MEANINGS BEYOND "RIGHT-WRONG" EVALUATIONS IN AN INTERACTIVE MANIPULATIVE ENVIRONMENT

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An important feature of the current reform movement in mathematics education is an appropriate use of computers at all grade levels. Yet economical constraints often stand in the way of incorporating technology into mathematics classrooms. This calls for technology-mediated pedagogy to shift the emphasis from resting solely on special purpose software as teaching and learning tools to a more sophisticated use of general computer tools commonly available across different educational settings.

The paper presents a spreadsheet-based environment for the study of the concept of percent which juxtaposes manipulative and computational features of the software. The tasks that structure this learning environment involve students' constructive activity within an interactive medium so that any action by a student leading to a qualitative change of an iconic input yields an immediate change of a numerical output. The pedagogy of the environment is structured by what French researches called addidactical situation —something that does not require explicit teacher intervention due to a careful design of the tasks allowing for a sufficient students' autonomy in a computer presence.

Another focus of the suggested computer activities is to encourage students' active involvement into construction of knowledge about the concept of percent by utilizing both univocal and dialogic functions of the environment. On the one hand, the univocality of the medium is concerned with an inert link between a student's numerical hypothesis regarding the percentage form of the iconic structure of a pictorial input and interactive evaluation of the hypothesis by the computer. On the other hand, the dialogic function of the medium is to generate new meanings of what may be wrong with such hypothesis through a follow-up manipulative on-computer activity in which a student constructs an iconic representation of the original guess. There is an assumption, rooted in the Vygotskian theory of semiotic mediation, that comparing a static configuration presented in a task to a dynamic structure actively constructed by a student enhances a transition from an initial erroneous guess to a correct answer by transforming a student's action into his or her consciousness.

The authors illustrate the above strategies by presenting results from a pilot study on students developing ideas about percentage in the computer environment.

# "PSYCHOLOGY" OF THE TI-92

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Technology such as the recently developed TI-92 hand-held "computer" has presented mathematics teachers and educators with numerous new educational opportunities for exploring and teaching mathematics. Such technology has also presented teachers and educators with a myriad of concerns regarding if and how it should be integrated into the mathematics curriculum. One objective for this discussion group is to raise awareness of and examine psychological issues associated with learning and teaching mathematics in a technological world. Another objective for the group is to grapple with the key question: What *is mathematics*, given the capabilities of the TI-92 and other technologies that now or soon will exist? The group organizers firmly believe that until this question is addressed by those involved with the teaching of mathematics and of inservice and pre-service mathematics teachers, appropriate integration of technology into the mathematics curriculum will remain an unattainable goal.

The group organizers will set a context for the discussion by briefly demonstrating the capabilities of the TI-92. Loaner calculators will be made available for those participants who want to gain additional hands-on experience. The demonstrations will be followed by 15 to 30 minutes of small group (3-4 participants per group) discussion of the question "*What is mathematics?*" After this discussion, the large group will reconvene to address the ideas presented in the small groups and to attempt to reach some consensus of response to the aforementioned question. Next, the group will brainstorm for 10 to 20 minutes on psychological issues associated with the integration of technology into the mathematics curriculum and on possible ways to address these issues through research, through the training and education of preservice and inservice teachers, and through the sharing of ideas with colleagues in the mathematics and mathematics education communities. The session will conclude with a discussion of potential follow-up activities for the group over the year following this meeting.

# STUDENT KNOWLEDGE

265

# ANALYZING STUDENTS' LEARNING WITH COMPUTER-BASED MICROWORLDS: DO YOU SEE WHAT I SEE?

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This paper describes a theoretical framework for analyzing students' learning with computer-based microworlds. The basic assumption is that students' ways of acting may be analyzed by coordinating social and psychological perspectives in an attempt to account for individual learning within the social context of the classroom. The goal of the analysis is to examine the different meanings that emerged during one interview conducted with a pair of third-grade students acting with a microworld that had been an integral part of their classroom instruction. The case documents the ways in which the two students, who appear to solve a problem together in the microworld, may be seen to be talking past each other. The analysis attempts to determine what each of the students saw and why they saw their activity in different ways

Computer-based microworlds have extended the possibilities for mathematical learning and research. For example, Kaput (1994) has argued that computer representations that are linked to real-world phenomena can enable students to bridge the islands of mathematical symbols and real-world experiences. The question for researchers is to determine if the students are seeing what the designers intend, or what the researcher sees, or even what another student may see. Addressing this question involves discerning when students are relying on advanced reasoning and when they are simply clicking and experimenting. To mitigate the possibility of misinterpretation, the approach taken in the following case study was to create an analytic framework that coordinates a social perspective of the context in which the students were acting with a psychological perspective of the students' possible mathematical conceptions (Cobb & Yackel, in press).

## Theoretical Framework

The theoretical framework that guided this analysis is based on the assumption that students' computer-based activity, mathematical interpretations, solutions, explanations, and justifications can be viewed as incorporating both psychological and social aspects (Cobb & Yackel, in press; Lampert, 1990; Simon, 1995; Voigt, 1995). To account for the interplay between these components, this framework attempts to take the individual acting with the computer *within the social microculture* as the unit of analysis. The psychological component of this study involved documenting students' individual ways of knowing and acting by exploring their conceptions of ten and hundred. The constructs underlying this analysis were based on the conceptual models of children's arithmetical learning developed by Steffe and his colleagues (Steffe, Cobb, & von Glasersfeld, 1988). At the risk of over-simplification, Steffe et al.'s framework was collapsed into three overarching constructs of place value. These include students' creation of: (1) numerical composites of ten and hundred, (2) composite units of ten and hundred, and (3) part-whole reasoning (For a detailed description of these distinctions, see Bowers, Cobb, & McClain, 1997).

The social component of this study involves documenting the social norms, the sociomathematical norms, and the mathematical practices in which the students engaged during in-class discussions and small-group activities (Cobb & Yackel, in press). The term mathematical practice is meant to convey the notion of taken-as-shared ways of constituting an activity that emerge as students attempt to explain and justify their explanations.

## Methodology

The case study presented in this paper is part of a teaching experiment that was conducted over 37 consecutive school days during the fall of 1994. The setting was a third-grade classroom in a public school in the southeastern part of the United States. The goal of the teaching experiment was to develop a reform-based instructional sequence and to research the ways in which the students developed increasingly sophisticated conceptions of place value and personally meaningful (but sometimes non-standard) ways to solve three-digit addition and subtraction problems. The teacher for this teaching experiment was an active member of the research and design team who replaced the regular classroom teacher for 45 minutes of math instruction each day. The decision to work with a "drop-in" teacher was based on the regular teacher's indication that her pedagogical beliefs were traditional in nature. For example, she chose to teach the students standard algorithms for adding and subtracting two- and three-digit numerals in the afternoons each day after the research team had left the school.

The candy factory instructional sequence and three accompanying microworlds were designed to support students' transition from acting in everyday situations involving grouping objects toward more formalized notions of place value. The instructional sequence was based on a scenario in which students invented ways to keep track of the inventory of a candy factory by packing sets of 10 candies into a roll and packing sets of 10 rolls into a box. Initial activities involved enumerating collections of candies and figuring out different arrangements of boxes, rolls, and pieces that could contain the same total quantity of candies (e.g., 543 candies could be packed into 5 boxes, 4 rolls, and 3 pieces or it could be packed into 4 boxes, 14 rolls, and 3 pieces, etc.) by drawing pictures and later using the first two microworlds. Analyses of the mathematical practices revealed that, as students engaged in these initial activities, they developed notions of how transformations conserve quantity. The final activities involved adding and subtracting quantities of candies by conducting transactions. These activities were augmented by the use of the third microworld in both whole-class and computer-laboratory settings.

The three microworlds each featured icons resembling students' drawings of boxes, rolls, and pieces. These icons could be packed and unpacked by clicking and dragging. The third microworld (shown in Figures 1 & 2) also included icons indicating candies that had been added or subtracted, and an inventory form that was linked to these storeroom icons. This linked representation system enabled students to initiate changes by clicking on the storeroom icons, in which the case the numerals on the inventory form would be updated, or by clicking on the dynamic numerals on the inventory form in which case the storeroom icons would be updated. It is critical to note that the goal of this linked representation was *not* to have the students attempt to figure out the supposed link between the two systems, but rather to provide an environment in which the students could work flexibly between the two representation systems to support their development of imagery for conserving quantity.

## Interview

The interview began as Martine and Carolyn attempted to solve the following problem:

> *There are 504 candies in the storeroom. A customer places an order for 69 candies. How many candies are left in the storeroom after the order is filled?*

Martine attempted to solve the problem using pencil and paper by writing:

$$\begin{array}{l}5\,0\,4\\6\,9\\\hline 1\,9\,4\end{array}$$

. Based on this and several other classroom episodes, it appears that, for Martine, solving pencil and paper tasks involved participating in the mathematical practice of following a procedure for manipulating digits which was most likely rooted in his experiences during afternoon classes with the regular classroom teacher. When the interviewer asked Carolyn if she agreed with Martine's answer, she stated, "When he wrote that down um, he said that there were 600, 6 boxes, and on the sheet it says they only made 69 pieces, so, he should have moved the 6 over..." Even though Martine never actually *said* "6 hundred" or "6 boxes," Carolyn interpreted his notation as indicating a written communication. Carolyn's explanation did not appear to be based on a memorized (but potentially meaningless) symbolic convention. Instead, she appeared to be reasoning symbolically, which is consistent with analyses of her post-interview activity. For her, the symbols had numerical significance and transcended any particular social context (i.e., she was not thinking of acting in the candy factory or acting in the afternoon instruction sessions). Upon hearing Carolyn's response, Martine reorganized his work as follows:

$$\begin{array}{r} 5\,0\,4 \\ -\;\;6\,9 \\ \hline 5\,6\,5 \end{array}$$

At this point, the interviewer asked Carolyn if she agreed with Martine's answer and if the two would like to work with the microworld. Carolyn indicated that she did not feel comfortable correcting Martine's work, and that the computer offered a more attractive alternative. As the two students began working in the microworld, they entered 5 boxes and 4 pieces as shown in Figure 1, Carolyn suggested that they unpack one box and one roll so that they could send out 6 rolls and 9 pieces. Her anticipatory statement indicates that she was thinking in terms of part-whole relations. In contrast, Martine, suggested that the pair unpack 7 rolls to send out 69 pieces. This suggestion, which was consistent with his interpretations during several prior whole-class discussions, was logical from Martine's point of view and indicates his structuring of the order to be sent out in terms of numerical composites rather than composite units. While one might argue that this interpretation is consistent with the way the problem was posed, it was not consistent with the taken-as-shared conventions that had been negotiated in prior classroom episodes. It is interesting to note that, due to the social norms that had been negotiated during whole-class discussions and as students worked in pairs in the computer lab, neither Carolyn nor Martine attempted to make sense of the other's interpretation from a mathematical point of view. For example, neither asked for clarification or turned to the computer to explore what would happen if 7 rolls were unpacked (which would have been easily accommodated by the software). Instead, Carolyn attempted to justify her approach by unpacking one box and one roll. As she did so, she stated, "You need to unpack only *one* roll. We are trying to figure out what is in the storeroom after we send out the ... umm ...[order]." Once Carolyn unpacked the box and roll, the screen appeared as shown in Figure 2.

After some further negotiation, Martine agreed to send out 6 rolls and 9 pieces. While this act may indicate that Martine reorganized his interpreta-

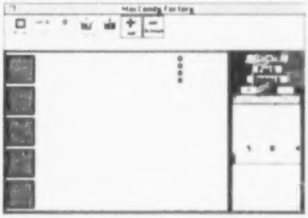

**Figure 1** *Initial screen showing 5 boxes and 4 pieces.*

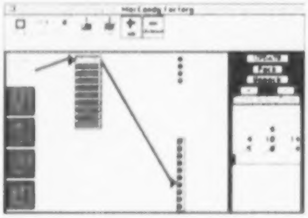

**Figure 2** *Screen appearance after one box was unpacked.*

tion of the 69 pieces so that he now "saw" it as a composite unit composed of 6 rolls and 9 pieces, it does not indicate that he was now reasoning in terms of part-whole relations or that he "saw" the results of his activity prior to acting. In fact, close examination of the social relations between the two indicates that Martine may have acquiesced to Carolyn without fully understanding the underlying mathematical concepts. In this way, Martine was "playing the school game"–a role with which he was very familiar. To complete the solution, Martine counted the remaining boxes, rolls, and pieces and recorded these totals on the inventory form.

## Discussion

As stated earlier, the underlying assumption of this analytic approach is that students' interpretations of their computer-based activity are constrained and enabled by their own mathematical conceptions and their interpretation of the social practices in which they are engaging. Based on this assumption, the two goals of this analysis are to document the nature of the differences between the students' interpretations of their computer-based activity and to address the ways in which the students' collective interpretations differed from the designers' intentions. In general, the microworlds were designed to support students' efforts to symbolize their activities and to generalize their situated activities by exploring relations between large numbers. For example, the designers assumed that students would pursue "what if..." questions by unpacking large numbers of boxes or rolls. In contrast, the class did not constitute the software as an exploratory medium. Instead, some, such as Cassie, interpreted their activities in terms of acting with an expressive medium. Others, such as Martine, viewed their activity in terms of acting with a simulation world. One difference between these two interpretations is that students who viewed their activities in terms of symbolizing were able to generalize their activities while others such as Martine who viewed their computer-based activities in terms of acting in a real (micro) world did not. It appears that Martine did not view the dynamic inventory form as a way to enact changes on the graphics because such a device is inconsistent with his interpretation of the microworld as an empirical simulation world in which one acts as one would in real life. That is, in real life, it is not possible to act on numbers to change objects; one acts on objects and then records the results with numbers. This interpretation is consistent with his interpretation of the boxes, rolls, and pieces as numerical composites. For him, acting in the microworld *was* acting with the boxes, rolls, and pieces, whereas his paper and pencil activity appeared to be unrelated to any real-world task.

In contrast, Carolyn, and many others in the class, had generalized their situated activity in terms of part-whole relations such that, for them, acting with the numerals, the icons, and pencil and paper all referred to the same conceptual imagery. It is critical to note that this generalization process did not arise from efforts to figure out the link between the numerals on the inventory form and the icons in the microworld. Instead, for Carolyn, the linked representation system was self-evident, and she was comfortable enacting transactions by clicking on the numbers on the inventory form or clicking on the graphics. The critical distinction between Carolyn's interpretation and that of Martine is that her efforts to anticipate the results of her acting were implicit in her decisions. In contrast, for Martine, who had not generalized his situated activities, the link between the notation systems was not apparent, nor was it an integral part of his situated activities.

Based on this analysis, one possible revision to the microworld would be to eliminate the linked representation system. Activities with the revised microworld would involve having students conduct transactions and transformations with the graphics, but develop their own ways to record their actions. As was seen in this class, many students attempted to curtail their drawing activity by recording their activity with numerals. In this way, the use of numbers arises as a natural way to expedite the symbolizing process. This approach parallels the historical development of symbolizations and would support students' development of meta-representational knowledge (cf. diSessa et al., 1991).

## References

Bowers, J., Cobb, P., & McClain, K. (1997, April). *Documenting the evolution of individual and collective mathematical activity in one third-grade classroom. Paper presented at the annual meeting of the American Educational Research Association, Chicago.*

Cobb, P., & Yackel, E. (in press). *Constructivist, emergent, and sociocultural perspectives in the context of developmental research. Educational Psychologist.*

diSessa, A. A., Hammer, D., Sherin, B., and Kolpakowski, T. (1991). *Inventing graphing: meta-representational expertise in children. Journal of Mathematical Behavior*, 10, 117-160.

Kaput, J. (1994). *The representational roles of technology in connecting mathematics with authentic experience. In R. Biehler, R. W. Scholz, R. Sträb, and B. Winkelmann (Eds.), Didactics of mathematics as a scientific discipline* (pp. 379-397). Dordrecht: Kluwer Academic Publishers.

Lampert, M. (1990). *When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. American Educational Research Journal, 27(1), 29–63.*

Simon, M. A. (1995). *Reconstructing mathematics pedagogy from a constructivist perspective. Journal for Research in Mathematics Education, 26, 114-145.*

Steffe, L. P., Cobb, P., & von Glasersfeld, E. (1988). *Construction of arithmetical meanings and strategies.* New York: Springer-Verlag.

Voigt, J. (1995). *Thematic patterns of interaction and sociomathematical norms. In P. Cobb & H. Bauersfeld (Eds.), Emergence of mathematical meaning: Interaction in classroom cultures (pp. 163-201).* Hillsdale, NJ: Lawrence Erlbaum Associates.

# WARNING: ASKING QUESTIONS MAY LOWER YOUR MATHEMATICAL STATUS IN SMALL GROUPS

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When students are placed in small groups to work on mathematics, how do they determine the mathematical status of each group member? In this report, mathematical status refers to the perceived trustworthiness of the contributions of individuals. Data gathered in this study suggest that one way a student may lose mathematical status is through asking many questions. A student may gain mathematical status through answering questions. Once the relative status of each group member is established, students with high mathematical status, "helpers", also ask teacher-type questions of other group members, the "helped", to check on their understanding. Even though this work is limited by the number of groups examined, it suggests several important areas for additional study. Do students distinguish between procedural questions, conceptual questions, and exploratory or extension questions in assigning status? Do students distinguish between real questions and veiled suggestions? What relationships exist between students' mathematical status and their understanding of mathematics?

As more students are placed in small groups to work on mathematics, the interactions in groups take on new importance. When students are placed in groups, they must assess the "trustworthiness" of each student's contributions. If students are unknown to each other, as often happens in today's classrooms, how do they judge the trustworthiness or mathematical status of group members? According to many researchers, being good at mathematics is associated with certainty, with getting the "right" answer quickly, and with remembering the rules. (Schoenfeld, 1985, Stodolsky, 1988). Lampert (1991) argues that these beliefs are shaped through many years of school experience and are deeply embedded in students' understandings of the classroom. How students assign mathematical status to their peers is not clear, but it is particularly important in light of the call for increased communication in mathematics classrooms as one basis for instruction. (NCTM, 1989). This report examines the differences in mathematical status of individuals in small groups in a college pre-calculus class. The larger scope of the study is to examine interactions within groups as they work on textbook problems and on laboratory explorations in hopes of better understanding what mathematics students come to know through peer interactions, and how they learn it. This report focuses on a narrow part of the larger study. Specifically, what evidence exists of differing mathematical status for members of a group? Is the involvement of individuals in the work of the group related to their status?

## Small Groups, Engagement, and Questions

Prior research has examined difference in interactions between high and low achieving students (Paradis & Peverly, 1994; Farivar, 1992; Swing & Peterson, 1981) and between teachers and groups of differing abilities (Cooper, 1981). Researchers have also examined the interactions between teachers and whole classes (Cobb, Wood, & Yackel, 1991; Lampert, 1991) and between teachers and small groups (Wood & Yackel, 1990). But little work has looked at the interactions among students within small groups without the teacher present.

A significant point to examine in small group interactions is the engagement of the individual group members. Engagement in mathematics has been linked to increased retention of at-risk students (Finn, 1993) and underrepresented groups such as women (Sax, 1993; McDermott, 1983). Steele (1993) reports that students are more likely to engage in mathematics during exploratory activities. Ivey & Williams (1993) report that many students engage in mathematics only if they see a place for their ideas and opinions in the work. These reports suggest that some students need to be valued for their unique contributions in a group in order to engage in mathematics. It seems reasonable that if a student is valued more highly in a group, then the student is more likely to engage in the mathematics class. Increased engagement should lead students to richer understandings.

This paper suggests one way in which students may assign or confirm mathematical status in small groups. The status of "helped", or poor student, may be established through asking many questions early in the group's existence, while the "helper", or good student, is the one who responds. Walen (1994) describes students' perceptions of questions in whole class discussions. She reports that students assessed the level of questions asked and labeled them as "dumb" or "good" questions. These labels were passed from the question to the questioner. The students in her study, however, were from a small school system and had known each other for many years. Thus the status given to the question may have come from a prior status given to the student. In this study, students who are unknown to each other also developed attitudes about questions and questioners, but the key factor appears to be quantity of questions early in the term.

## Being in the World of the Classroom

Analysis of data proceeded from a framework that assumes that students bring many assumptions and attitudes into the classroom. (For a more complete exposition of this framework, see Williams, 1993.) Briefly, this framework assumes that students' actions reflect the Heideggerian notion of being-in-the-world. They are thrown into a situation, and they act. How they judge a situation and act within that situation is often at a non-reflective level. As long as their assumptions do not come into conflict with something within the situation, they do not examine the basis of their actions. Precisely because these assumptions are not reflected upon, their existence has profound impact on the actions of individuals. At the same time, the underlying nature of the assumptions make them difficult to articulate. If I ask student A whether student B, in the same group, is smart or not smart, student A will have an answer. If I ask student A about the basis for that assessment, the answer is likely to be vague. However, by looking at students' actions and interactions, some indication of their underlying assumptions can often be inferred.

The data presented in this report come from videotapes of small group interaction during one semester of pre-calculus taught by the author at a comprehensive regional university. After obtaining informed consent, the class was videotaped each time it met. Each day, established small groups reviewed assignments or performed laboratory explorations together. Each of three groups was videotaped alternately beginning on the third day of class. Due to length constraints, this paper will report on the evidence of differing mathematical status within groups as a whole, but will use one of the groups as an example. This focus group consisted of three males, whom I will call Paul, Steve, and Ronnie. This group was videotaped during nine, 75-minute classes spread through a 15 week term. The interactions discussed here took place without the teacher present.

## Suggestions, Questions, and Mathematical Status

As the term progressed, it became obvious that the groups valued the contributions of each member differently. One strong example of this discrepancy can be seen in the two lab activity tapes which were part of the nine videotapes of Paul, Steve, and Ronnie. The first of these two episodes occurred in late September, about one month after the beginning of classes, and the second occurred in December about a week before the end of the term. In the first lab, all of the students participated fully. They used a Calculator Based Laboratory™ connected to a TI-85 calculator to collect data on the path of a ball thrown up and allowed to fall freely. They took turns attempting to toss the ball over the detector, but Paul always operated the calculator. They all examined the resulting graph and collectively decided when their graph was "good enough." During this episode, there were no questions asked and only a few suggestions offered because the group did not have difficulty in producing or agreeing on the required graphs. All of the members were involved in deciding how to accomplish their tasks and when they were done. During the second lab, however, the interactions were different. In this lab, students were given a styrofoam cone and a knife, and they were to cut the cone to produce sections whose outlines were a parabola, one branch of a hyperbola, a circle, and an ellipse. This time, Paul and Steve discussed with each other what to do, while Ronnie looked over their shoulders. Once they reached a decision, they gave Ronnie the knife to make the cuts, but only after Steve had marked where to cut. After cutting the cone, Ronnie was teased about his poor job of cutting. Interestingly, as the students passed around the various pieces, it was Ronnie who asked which piece was the parabola and which was the hyperbola. Unlike the earlier lab where everyone seemed to participate more or less equally, in this lab, Ronnie was not included in the decision making part of the lab. It appeared that all three students were participating, but their participation was not of the same type.

To further document the disparity noted above, each group's tapes were examined. Using the constant comparative method, student suggestions were coded, and the reception of the group was noted. A pattern developed that substantiated the observation that some students' suggestions were seldom followed, while other students' suggestions were usually or always followed. During the first half of the term, Ronnie made about half of all oral suggestions in his group, with many of them ignored. Steve made most of the rest of the oral suggestions, and these suggestions were usually either implemented immediately or discussed between Paul and Steve. Paul was usually the physical center of this group's activity, so most of his suggestions were not oral. He would write something down on paper, if they were trying to perform some mathematical procedure, or he would key something into his calculator and show the resulting display to the others. Not surprisingly, students whose suggestions were not followed tended to offer fewer suggestions as the term progressed. During the second half of the term, Ronnie made fewer than ten percent of the oral suggestions. Steve continued to make about the same number of suggestions as earlier, but now they accounted for a much higher percentage of all suggestions. Paul also began to give more oral suggestions before or as he tried them.

Further examination of the groups' interactions led to the observation that the more questions a student asked early in the term, the fewer suggestions he or she made later in the term. Ronnie asked the greatest number of questions of the three students, but early in the term, he made about as many suggestions as he asked questions. Later, he continued to ask many questions, but he made fewer suggestions. Early in the term, all questions tended to be put to the group as a whole, while later the questions became more directed towards particular group members, giving further evidence of perceived differences in mathematical status. Of the questions clearly directed toward an individual (either by name or by touching to gain attention) at least sixty percent were directed toward Paul. In general, Ronnie would ask Paul most of his questions and would only ask Steve if Paul were busy. Paul directed all of his real questions toward Steve, and Ronnie had only three real questions directly addressed to him in all nine tapes. By real questions, I mean questions that ask for information unknown to the asker.

Toward the middle of the term, question asking changed. While the "helped" members still asked many questions, the "helpers" also began asking questions, but not real questions.

Paul: "OK, Ronnie, how do you get A?"

Ronnie: "Fill in another point?"

Paul: "Yeah, that's right."

These leading questions were similar in tone and type to questions usually asked by teachers (Mehan, 1979). The types of questions asked by the "helped" were also examined, revealing many procedural questions. A typical example was a question about the calculator.

Ronnie: (Touching Paul's arm) "How do you get the picture up?"

Paul: "Stat, Draw, Scatter." (key strokes on the calculator)

The interactions between "helpers" and "helped" in two of the three groups followed this pattern closely. The third group also followed this pattern, but to a lesser extent. One possible explanation may be that some members of the third group were known to each other prior to this class, unlike the members of groups one and two.

## Few Answers and Still more Questions

The level of involvement in the group appears to be closely linked to a student's status, and thus to a student's engagement in mathematics. This study suggests that question asking is one way that students may acquire or confirm the status of "helped" or "helper" in small groups. There is not sufficient evidence to support a causal link between question asking and mathematical status, but it is apparent that these two issues are linked. How quickly mathematical status is assigned cannot be determined from this study. This report does further our understanding of how students interact in small groups, and suggests one way in which a particular student may come to have high or low mathematical status in a group. It also raises an interesting question concerning gender issues in mathematics. If girls and women frequently offer suggestions in the form of questions, as suggested by various writers (beginning with Lakoff, 1975), does this practice lead to lower mathematical status? Other questions also emerge. Which comes first, mathematical status or questioning behavior? Do students distinguish between genuine questions, disguised suggestions, and teacher-like leading questions in assigning status? Do students distinguish between procedural questions, conceptual questions, and exploratory or extension questions in assigning status? What relationships exist between students' mathematical status and their understanding of mathematics? Like much research, this study suggests many new questions.

## References

Cobb, P., Wood, T., & Yackel, E. (1991). A constructivist approach to second grade mathematics. In E. von Glasersfeld (Ed.), *Radical constructivism in mathematics education.* Kluwer Academic: Dordrecht, The Netherlands.

Cooper, C. R. (1981). *An ethnographic study of teaching style.* Paper presented at the Midwest Regional Conference on Qualitative Research in Education, Kent, OH.

Farivar, S. (1992). *Middle school math students' reactions to heterogeneous small group work: they like it!* Paper presented at the Annual Meeting of the American Educational Research Association, San Francisco, CA.

Finn, J. D. (1993). *School engagement & students at risk.* (ERIC Document Reproduction Service No. ED 362 322)

Ivey, K. M. C., & Williams, S. R. (1993). *Investment of self as a component of mathematical engagement.* Paper presented at the Fifteenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Pacific Grove, CA.

Lakoff, R. (1975). *Language and woman's place.* New York: Harper and Row.

Lampert, M. (1991). Connecting mathematical teaching and learning. In E. Fennema, T. P. Carpenter, & S. J. Lamon, (Eds.), *Integrating research on teaching and learning mathematics* (pp. 121-152). Albany, NY: SUNY Press.

Mehan, H. (1979, October). "What time is it, Denise?": Asking known information questions in classroom discourse. *Theory into Practice, 18*(4), 285-94.

McDermott, M. (1983). *The impact of classroom interaction patterns on students' achievement-related beliefs and behaviors.* Paper presented at the Biennial Meeting of the Society for Research in Child Development, Detroit, MI.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics.* Reston, VA: Author.

Paradis, L. M. & Peverly, S. (1994). *The effects of knowledge and task on students' peer-directed questions in modified cooperative learning groups.* Paper presented at the Annual Meeting of the American Educational Research Association, New Orleans, LA.

Sax, L. J. (1993). *The development of mathematical self-concept during college: Unique benefits for women in math-intensive majors?* Paper presented at the Annual Meeting of the Association for the Study of Higher Education, Pittsburgh, PA.

Schoenfeld, A. H. (1985). Metacognitive and epistemological issues in mathematical understanding. In E. A. Silver (Ed.), *Teaching and learning math-*

*ematical problem solving: Multiple research perspectives* (pp. 361-379). Hillsdale, NJ: Lawrence Erlbaum Associates.

Steele, D. F. (1993). *What mathematics students can teach us about educational engagement: Lessons from the middle school.* Paper presented at the Annual Meeting of the American Educational Research Association, Atlanta, GA.

Stodolsky, S. S. (1988). *The subject matters: Classroom activity in math and social studies.* Chicago, IL: University of Chicago Press.

Swing, S. R., & Peterson, P. L. (1981). *The relationship of student ability and small-group interaction to student achievement.* Paper presented at the Annual Meeting of the American Educational Research Association, Los Angeles, CA.

Walen, S. B. (1994). *Identification of student-question teacher-question student-response pattern: Students' interpretation of empowerment.* Paper presented at the Annual Meeting of the National Council of Teachers of Mathematics. Indianapolis, IN.

Williams, S. R. (1993). Mathematics and being in the world: Toward an interpretive framework. *For the Learning of Mathematics, 13*(2), 2-7.

Wood, T., & Yackel, E. (1990). The development of collaborative dialogue within small group interactions. In L. P. Steffe and T. Wood (Eds.), *Transforming children's mathematics education* (pp. 244-252). Hillsdale, NJ: Lawrence Erlbaum Associates.

# OCCASIONING UNDERSTANDING: UNDERSTANDING OCCASIONING

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As part of our research of students' mathematical understanding we are exploring how interactions with the environment and those persons in it become occasions for student and teacher knowing. The illustration used in this paper comes from a study of a class of tenth grade mathematics students with weak performance histories in school mathematics as they worked in a hands-on and pre-formal algebra environment. In this paper, we discuss how the teacher's initial prompt became an occasion for the students' mathematical activity; how one student's actions become occasions for another student's learning; how the teacher's interaction with the students throughout the session was implicated in the students' mathematical knowing; and how the teacher's pedagogical-content knowing was revealed and changed in the interaction within this setting.

## Introduction

> [Heinz von Foerster] coined the phrase "order from noise" to indicate that a self-organizing system does not just "import" order from its environment, but takes in new energy rich matter, integrates it into its own structure, and thereby increases its order. (Capra, 1996, p.84)

Since human learners can be thought of as self-organizing systems *par excellence*, the above quote suggests that in mathematics classes the student is not directly instructed by the teacher, or materials, or text, but must use his or her own schemes to make sense of the environment and hence alter his or her mathematical understanding. At the same time the above quote suggests that the environment and others in it act as sources of "energy rich matter" and hence are fully implicated in the student's cognition. The question of how this works is at the centre of much contemporary research in mathematics education (e.g., Cobb, Yackel, Wood, 1992; Confrey, 1995; Simmt, Kieren, Gordon Calvert, 1996). The purpose of our paper is to continue the discussion by exploring the question: *How, through student (and teacher) actions, do aspects of the classroom environment, and interaction among students, and among students and teachers become transformed into occasions for mathematical knowing?*.

We consider this question by interpreting the interaction among tenth grade mathematics students with weak performance histories in school mathematics as they worked with a teacher in a hands-on and pre-formal polynomial algebra environment. In particular, we will illustrate how materials and the interactions raise the possibilities that are uniquely transformed by individual students using their own schemes into mathematical knowing actions. We will also illustrate how reciprocally such actions help change the teacher's pedagogical content knowledge.

## Theoretical Perspective

It is easy to think of mathematical activity as occurring in response to discrete stimulus settings or as response oriented problem solving. Such activity can be captured in the oral and written responses of students which can then either be matched and compared with pre-given "answers" or analysed in their own right for patterns and characteristics. But such a view of mathematical products and the study of them, however valuable and however complex, might be thought of as a study of the disembodied surface of mathematical knowing. The theoretical framework underlying this research asks us to think differently and to consider mathematical knowing as a fully embodied phenomenon. It is based on the biological, evolutionary and ecological work on cognition developed in Maturana and Varela (1987) and Varela, Thompson and Rosch (1991). Rather than observing mathematical cognition as answer-generating or even as problem-solving, this enactive view observes mathematical knowing in the personal action of bringing forth a world of mathematical significance with others in a sphere of behavioural possibilities (Maturana and Varela, 1987). In this paper we consider *occasioning* as a mechanism which helps explain how this "bringing forth" occurs. *Occasioning occurs as the person selects elements from the environment and acts on them, thus changing the environment and in fact bringing forth a world; but also the history of the person's mathematical activity is changed and thus his or her understanding (structure–schemes) is altered.*

## Methods and Data Sources

In order to study the way in which the environment and student actions and interactions allowed for the occasioning of changing understanding a mathematics class of 28 students participated in a two month long research project as they studied polynomial algebra. The authors, supported by the observation of and continuing interaction with the regular classroom teacher developed kits of 2 and 3 dimensional polynomial materials and some 60 lesson set-ups which were framed in the context of what was called the *Polynomial Engineering Project*. It was intended that these materials and lessons would allow the students to experience and use polynomial concepts, language, and computations to design and describe mathematical objects. The study was conducted across 25 80-minute class periods and followed the objectives of the provincial curriculum. Although the classroom teacher was present for all of the lessons and worked with small groups of students, it was one of the researchers that took responsibility for teaching the class over the course of the research project. In addition to video and audio taping 1 or 2 groups of students each day and interacting and interviewing many of the students on an ongoing basis throughout the unit, two pairs of students participated in a clinical interview five months after the classroom teaching experiment. Student work, observer field notes, video and audio tapes, interview records, notes from interaction with the classroom teacher and the input of the researchers who participated in the activity at the research site formed the corpus of data for this study.

## Results and Interpretations

The interpretations developed below which attempt to answer the central question posed in this paper are based on the interactions of two students, Donny and Jennifer as they worked with the teacher/researcher, Tom Kieren, on a variable-entry prompt. This prompt and the actions of the students discussed here were typical of the ways in which many of the students in this class came to think of and treat polynomial algebra. When Donny was asked why he was in the non-academic mathematics class, he, like others in his class had been heard to say, replied "Because I suck at mathematics." In contrast, Jennifer responded that she was "LD" (learning disabled) and had always been placed in such classes. Records of the students' work (figure 1) and the vignette (figure 2) that follow are based on a short portion of a clinical interview in which Donny, who "sucked at math," and Jennifer, who was "LD" were exploring polynomial factors and products.

In order to provide a response to the research question posed in this paper and remain within its scope, the interpretation below takes the form of brief answers to various sub-questions.

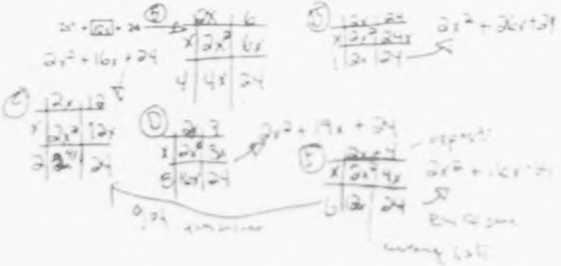

Figure 1 *A part of Jennifer and Donny's working records.*

*1) How does the prompt come to provide occasions for learning?* In light of the role Donny plays in the dialogue in Figure 2, one would think that he would have immediately made sophisticated responses to the prompt. But such was not the case. Donny was puzzled by the missing term and even how to interpret the "24". His initial response was to take out the polynomial kit and build "rectangles." Jennifer responded differently. Because she had an image of factorable rectangles (that is, rectangles for which the length and width could be expressed as first degree polynomials) and had come to associate that image with the mathematics of the grid, she used this image and this "tool" to create a variety of possible polynomials. This difference, as well as the profusion of polynomials which were produced by Jennifer and Donny, points to the way in which the prompt was turned into occasions for mathematical action. The prompt did not cause a particular set of actions; rather, the students acted within the mathematical constraints of the prompt and the constraints of their own histories (Jennifer with grids and Donny, at first, with materials).

*2) How does the action of one student come to provide an occasion for other students' learning?* In lines 5-10 we see the effect of Jennifer's polynomial creations on both the teacher and Donny. By using different factors of 24 to create binomial factors (see Figure 1), Jennifer provided instances which surprised the teacher and prompted Donny to reflect on the products of his, Jennifer's, and the teacher's interacting activities.

As discussed above, Jennifer's image of factorable polynomials allowed her to use grids to generate polynomials. But her collection of polynomials became a continuing source inspiring Donny's pattern noticing and pattern creating activities. For example, Donny suggested that they turn their attention to what happens when you "reverse" the factors of 24 (e.g., $2x + 4$ and $x + 6$ compared with $2x + 6$ and $x + 4$). Donny and the teacher became very interested in this idea; but their very interest and prompting brought Jennifer's technical and reflective attention to this task as well. Notice in lines 15-17 she adjusts her technique and in line 18 she now offers a conjecture. Thus Jennifer's actions (and the teacher's) enlarged the sphere of possibilities for Donny. However, Donny's thinking prompts not only different actions on Jennifer's part (lines 16, 26) but different reflections as well (see line, 16 18). Thus we might say Donny's and Jennifer's mathematics in action are co-implicated; their actions co-emerge.

*3) What is the impact of the teacher on the occasions for learning in this setting?* Although the teacher provided the initial prompt for this lesson, he did not "cause" the learning nor did he simply facilitate or model that learning. It is clear that the teacher was working *with* the students as they worked. This is illustrated in his indications of surprise and delight in the mathematics as it developed in the actions of the students (e.g., lines 7, 25). Through his own real interest in the situation he provides other possibilities or in von Foerster's terms

In this scene, Donny and Jennifer were responding to the prompt:

*The following polynomial is known to form a rectangular design, however it is missing a term. Before you can pass on the order to your client you need to find out what the missing term might be. Offer a possibility or a list of possibilities to go over with your partner.*

$2x^2 + \quad + 24$

Jennifer had just found that by using 3 and 8 (factors of 24 that Donny had suggested they consider) that she could generate another possible polynomial.

"Oh! That is a pretty good one. I didn't know it was going to turn out quite the way it did." noted Tom, the teacher/researcher.

"I didn't even see that." responded Donny.

"Because you were talking too much," Jennifer, replied as she looked up from her sheet.

Donny had been generating factors of 24 out loud. As he did so he looked to Jennifer to compute the polynomials that were generated using the set of factors he had identified. By this time it had struck Donny that there was some potential in taking the same two factors of 24 and reversing them in the grid that Jennifer had been using to do her computations.

"Now try your reverse," Tom suggested.

"It's going to be the same." said Jennifer. "You mean put a 2x here and then—"

"No," replied Donny. "Put x and then switch the 4 and the 6."

Jennifer was still not sure why Donny and Tom were suggesting she do this. "You are still going to get the same thing."

As she carried out the computation, her records were visible for Tom and Donny to reflect on. "What's the difference," Donny asked.

"There it is," said Tom as he pointed to the position of the x terms on the grid. "It's the way it looks. They look quite different."

But Donny noticed something else. "No, not only that, there's no 12 here and there's a 12 here."

"Oh, right." Tom's voice reflected his surprise.

"So—"

"Oh, they are different. We are getting this one." Tom pointed to another polynomial product that had been calculated previously.

By this time Tom and Donny were very excited. Daniel asked Jennifer to do another one.

Tom was still working through what he and Donny had noticed. "So you are saying actually when you had 2x + 4 and x + 6 you got the same shape down here as this one."

"Right." Donny confirmed Tom's observation.

"Wow, this is just incredible. I didn't know this."

"And the only thing that is different is right here." Donny concluded.

Donny, Jessica, and Tom continued to work on this property that switching the factors of 24 between the c and the 2x terms would generate polynomial products that they found from other factors.

When a third set of factors of 24 generated yet another new polynomial product Donny blurted out. "That actually scares me. Just by simply putting it in a different part and you get a different answer. You know your work, but you just notice we get different answers. You put that on the test and you'll get kids contradicting themselves saying, is this really right?"

**Figure 2** *A vignette created from the transcripts and video tape of a pair of students, Donny and Jennifer, and the teacher/researcher, Tom, responding to a variable-entry prompt*

"energy rich matter" for the students as he points to what he sees as interesting in the mathematics they are bringing forth together (e.g., lines 15, 22-23, 27). As is perhaps best seen in Donny's remarks in lines 37-40, this matter is used by the students in their own way not only to generate mathematics but to reflect on how they are doing this.

*4) What is the nature of the teacher's pedagogical content knowledge as revealed in this setting?* It is typical to think of a teacher's pedagogical content knowledge as a blending of content and pedagogy (Shulman, 1986) into how a particular topic (e.g., factorable polynomials) is organized and represented for instruction (e.g., *Polynomial Kits*, grid techniques, even variable-entry prompts). In addition, pedagogical content knowledge might refer to teacher's mental constructions of the relationship between student's learning (perhaps as measured on a test) and pre-given mathematical knowledge. Our interpretation of the vignette in figure 2 adds a lived dimension to pedagogical content knowledge . Pedagogical content knowledge is not something that the teacher acquires. Like the student's mathematical knowing, pedagogical content knowing is a bringing forth of a world of significance with others, especially with students in the course of mathematical knowing in action and interaction. It is the students' actions in the environment which actually occasion the transformation of the teacher's pedagogical content knowledge (lines 24-30). Here the teacher is changing what he knows pedagogically about polynomials based on the mathematical concerns of the students. When he says "I didn't know this" in line 33, the teacher is not revealing ignorance of factorable polynomials per se. What he didn't know and what he was in the process of learning was how this might manifest itself in the mathematical lives of students. Thus the actions of the students provided the possibilities for teacher learning not just about student thinking but about the way certain polynomial ideas can occur; that is, new to him pedagogical mathematics. Here the teacher was the learner and in certain places (e.g., lines 20-34) this reciprocity of teaching and learning is especially evident.

## Summary

This paper has been about looking at mathematics teaching/learning not simply as problem solving but as bringing forth a world of significance with others. We have argued that such a view prompts us to think of *how elements of the environment and interactions in it are transformed by the knowing actions of individuals into occasions for learning*. If in such a setting the teacher acts to open possibilities for the learners and vice versa, "energy rich matter" can be transformed by both teacher and students together into occasions for richer mathematics in action.

## References

Capra, F. (1996). *The web of life.* New York: Anchor Books Doubleday.

Cobb, P., Yackel, E., Wood, T. (1992). Interaction and learning in mathematics situations. *Educational Studies in Mathematics, 23*, 99-122.

Confrey, J. (1995). A theory of intellectual development: Part III. *For the Learning of Mathematics, 15*(2), 36-45.

Maturana, H., & Varela, F. (1987). *The tree of knowledge: The biological roots of human understanding* (rev. ed.). Boston: Shambhala Publications Inc.

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Simmt, E., Kieren, T., & Gordon Calvert, L. (1996). All at once: Understanding and interaction in mathematics. In E. Jakubowski, D. Watkins & H. Biske (Eds.), *Proceedings of the Eighteenth Annual Meeting North American Chapter of the International Group of the Psychology of Mathematics Education Vol. 1* (pp. 173 - 177). Columbus: The ERIC Clearinghouse for Science, Mathematics and Environmental Education.

Varela, F., Thompson, E., & Rosch, E. (1992). *The embodied mind: Cognitive science and human experience.* Cambridge: MIT Press.

# AN ANALYSIS OF STUDENTS' DEVELOPMENT OF REASONING STRATEGIES WITHIN THE CONTEXT OF MEASUREMENT

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The purpose of this paper is to document the interactive process by which students developed personally-meaningful ways to reason mathematically within the context of measurement. This process encompasses the proactive role of the teacher, the contribution of carefully sequenced instructional activities, and the importance of discussions in which students explain and justify their thinking. In doing so, we will present episodes taken from a four-month teaching experiment conducted in a first-grade classroom. Our intent is not to offer examples of exemplary teaching. It is instead to provide a context in which to examine the relationship between the role of measurement and students' construction of increasingly sophisticated ways to think and reason mathematically.

## Introduction

Our purpose in this paper is to describe how one group of students developed personally-meaningful ways to reason mathematically within the context of measurement. To clarify our viewpoint, we present episodes taken from a first-grade classroom in which we conducted a four-month teaching experiment. One of the goals of the teaching experiment was to develop instructional sequences designed to support first-graders construction of meaningful understandings for (1) measurement and (2) mental computation and estimation strategies for numbers up to 100. A primary focus when developing the instructional sequences was to support students' multiple interpretations of problem situations. These interpretations would then serve as the basis for classroom discussions in which students explained their mathematical reasoning. Our intent in presenting the episodes is not to offer examples of exemplary teaching. It is instead to provide a setting in which to examine measurement as a context for supporting students' construction of sophisticated ways to think and reason mathematically.

The first-grade classroom in the study is of particular interest because preliminary analysis of the data indicates that the students' development of mathematical reasoning was substantial. For example, students who at the beginning of the classroom teaching experiment were unable to reason quantitatively with numbers up to twenty were by the end of the experiment able to use flexible strategies for mental computation with two-digit numbers. The particular approach we take to analyzing students' mathematical activity involves coordinating constructivist analyses of individual students' activities and meanings with an analysis of the communal mathematical practices in which they occur (cf. Cobb & Yackel, in press).

## Data Corpus

Data were collected during the spring semester of the 1996 school year and consist of daily video-recordings of 62 mathematics lessons from two cameras. One camera focused on the white board and students who came to the front of the room to offer explanations and justifications. The second camera focused on the students seated in whole class. Additional documentation consists of copies of all the students' written work, three sets of daily field notes that summarize classroom events, notes from the teacher's daily planning and reflection with the research team, field notes and audio tapes from weekly project meetings, and taped clinical interviews conducted with each student in January and May.

## The Instructional Sequences

As we have noted, a primary focus of the teaching experiment was to develop two instructional sequences. The first sequence dealt with measurement and the second one built on the measuring activities to support students' construction of mental computation and estimation strategies for reasoning with numbers to 100. In the case of the instructional sequence that dealt with measurement, our initial goal was that students might come to reason mathematically about measurement and not merely measure accurately. The focus was therefore on the development of understanding rather than the correct use of tools. In particular, we hoped that the students would come to interpret the activity of measuring as the accumulation of distance (cf. Thompson & Thompson, 1996). Further, our intent was that the results of measuring would be structured quantities of known measure. We therefore hoped that the students would come to act in a spatial environment in which distances are structured quantities whose numerical measure can be specified by measuring. In such an environment, it would be self evident that while distances are invariant quantities, their measures vary according to the size of the measurement unit used.

As we will see, measuring with composite units became an established mathematics practice in the course of the teaching experiment. Initially, the students drew around their shoes and taped five shoe-prints together to create a unit they named a *footstrip*. Later, in the setting of an ongoing narrative about a community of Smurfs, the students used a bar of ten unifix cubes to measure. These instructional activities evolved into measuring with a strip that was the length of 100 unifix cubes. This in turn made it possible for the students' activity of measuring to serve as the starting point for the second instructional sequence that focused on mental computation and estimation with two-digit numbers. In terms of Greeno's (1991) environmental metaphor, the intent of this latter sequence was that the students would come to act in a quantitative environment structured in terms of relationships between numbers up to 100. We, therefore, focused on students' construction of numerical relationships that are implicit in their calculational methods. This shifts the focus from calculational strategies per se to the mathematical interpretations and understandings that make flexible use possible.

## Classroom Episodes

The instructional activities used in the teaching experiment were typically posed in the context of an ongoing narrative. To accomplish this, the teacher engaged the students in a story in which the characters encountered various problems that the students were asked to solve. The narratives both served to ground the students' activity in imagery and provided a point of reference as they explained their reasoning. In addition, the problems were sequenced within the narratives so that the students developed increasingly effective measurement tools with the teacher's support. Further, the narrative supported the emergence of tools out of students' problem solving activity.

The second narrative developed during the measurement sequence involved a community of Smurfs who often encountered problems that involved finding the length or height of certain objects. The teacher explained that the Smurf's decided to measure by stacking cans the height of the object to be measured. In the classroom, the students used Unifix Cubes as substitutes for cans and measured numerous objects for the Smurfs such as the height of the wall around the Smurf village, the length of the animal pens, and the depth of the water in the river. After several measuring activities, the teacher explained to the students that the Smurfs were getting tired of carrying around the large number of cans needed for measuring. The students agreed that this was cumbersome and discussed alternative approaches. Several suggested iterating a bar of cubes (cans) that they eventually called a *Smurf bar*.

When measuring with the Smurf bar, all the students measured by iterating the bar along the length of the item to be measured and counting by tens. However, some counted the last cubes of the measure within the last iterated decade. Solutions of this type became the focus of discussions as can be seen in an incident that occurred two weeks after the measurement sequence began. The teacher had posed the following task: *The Smurfs are building a shed. They need to cut some planks out of a long piece of board. Each plank must be 23 cans long. Show on the board where they would cut to get a plank 23 cans long.* Students had been given long pieces of adding machine tape as the board and asked to use the Smurf bar to measure a plank the length of 23 cans. Ann was the first student asked to share her solution process with the class. She showed how she had measured a length of 23 cans by iterating the bar twice and then counting 21, 22, and 23 *beyond* the second iteration. When she finished, Eddy disagreed stating that Ann had shown a distance of 33 since he counted cubes 21, 22, and 23 *within* the second iteration, thus measuring a length that was actually 13 cubes. Ann then asked Eddy to explain his answer again.

Eddy: Ten (places bar down in first iteration), 20 (moves bar to second iteration). (Pause) I changed by mind. She's right.

Teacher: What do you mean?

Eddy: This would be 20 (points to end of second iteration).

Teacher: What would be 20?

Eddy: This is 20 right here (places one hand at the beginning of the "plank" and the other at the end of the second iteration). This is the 20.

Teacher: So that where your fingers are shows a plank that would be 20 cans long? Is that what you mean? Any questions for Eddy so far?

Eddy: Then if I move it up just three more. There. (Breaks the bar to show 3 cans and places the 3 cans beyond 20). That's 23.

It appeared that in the course of re-explaining his solution, Eddy reflected on Ann's method and reconceptualized what he was doing when he iterated the bar. Initially, for Eddy, placing the Smurf bar down the second time as he said "20" meant the twenties decade. Therefore, for him, 21, 22, and 23 lay within the second iteration. However, he subsequently reconceptualized "20" as referring to the distanced measured by iterating the bar twice and realized that 21, 22, and 23 must lie beyond the distance whose measure was 20. This type of reasoning was supported by Eddy at least implicitly *folding back* (McClain & Cobb, in press) to counting the cans that he iterated when moving the bar (i.e., the measure of the first two iterations was 20 because he would count 20 cans).

It is important to note that the teacher's overriding concern in this episode was not to ensure that all the students measured correctly. In fact, the teacher frequently called on students who had reasoned differently about problems in order to make it possible for the class to reflect on and discuss the quantities being established by measuring. Her goal was that measuring with the Smurf bar would come to signify the measure of the distance iterated thus far rather than the single iteration that they made as they said a particular number word. Her focus was therefore on the development of mathematical reasoning that would make it possible for the students to measure correctly with understanding.

After the students had measured several planks and other items with the Smurf bar, the teacher explained that the Smurfs decided they needed a new measurement tool so they would not have to carry any cans around with them each time they wanted to measure. In the ensuing discussion, students proposed creating a paper strip that would be the same length as a Smurf bar and marked with the increments for the cans. During a discussion about the meaning of measuring by iterating a ten-strip, the teacher taped several of the students' strips end-to-end on the white board to show successive placements of the strip. In doing so, she created a *measurement strip* 100 cans long. The transition from the measurement sequence to the mental computation and estimation sequence occurred when the students began to use the measurement strip to reason about the relationship between the lengths or heights of objects that were not physically present. One of the first instructional activities in the mental computation and estimation sequence involved an experiment the Smurfs were conducting with sunflower seeds. The teacher explained that the Smurfs typically grew sunflowers that were 51 cans tall. However, in one of the experiments, the sunflowers grew only 45 cans tall. Students were then asked to find the difference in heights and were given only a measurement strip. As a consequence, they could not create *objects* 45 cans and 51 cans long to represent the sunflowers, but instead had to reason with the strip.

Students first worked in pairs to solve the task and then discussed their solution in the whole-class setting. The teacher began by placing a vertical measurement strip on the wall and asking students to mark both 51 cans and 45 cans (see Figure 1). An issue that emerged almost immediately in the discussion was that of whether to count the lines or the spaces on the strip.

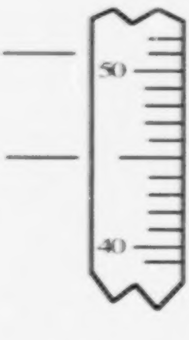

**Figure 1** *Measurement strip marked to show 51 and 45.*

Pete: Here (points to 51) all the way down to here (points to 45) would be seven.

Teacher: Can you show me the seven?

Pete: Here is 51 and here is 45 and here is 1, 2, 3, 4, 5, 6, 7 (points to lines as he counts).

Pat: I have a question. You are supposed to count the spaces not the lines.

Teacher: Why, Pat?

Pat: The cans of food are bigger than the lines and you are trying to figure out how many cans not lines.

Teacher: So when you say space you think of this space as a can of food (points)?

Pat: And that's how much and you're trying to figure out how much that is.

For Pat, reasoning with the measurement strip was related to the prior activity of measuring with a Smurf bar. As a consequence, the spaces signified unifix cubes/cans for him. In contrast, reasoning with the strip did not appear to be grounded in prior activity for Pete and he was simply trying to figure out a way to use it to solve the task at hand. However, Pat's explanation led Pete to modify how he reasoned with the strip. This is evidenced by the fact that Pete asked if he could use a can/cube to help him solve the task. He then folded back to prior activity by placing a single cube on the measurement strip and iterating the spaces between 45 and 51 to arrive at the answer of six rather than seven.

Immediately after the exchange between Pat and Pete, Andy gave an explanation that involved reasoning about the quantities in a different way.

Andy: If you went from 50 down five you'd get to 45 cans. Think 5 less than 50. But you are really one more so it's six since it's one more than 50.

Andy's explanation indicates that for him, as for Pat, 45 and 51 signified distances from the bottom of the strip measured in cans. The task for him was to find the difference between these two quantities, and he did so by reasoning with the strip. We would in fact argue that the strip supported the shift he made from a counting to a thinking strategy solution in which he reasoned that 50 to 45 was five, so 51 to 45 is six.

It is important to note that the solution method offered by Andy fit with the teacher's pedagogical agenda of supporting students' development of increasingly sophisticated strategies. However, the teacher was also aware of differences in her students' reasoning and did not want to create a situation where students simply imitated strategies that they did not understand. As a result, she continued to acknowledge the differing ways that students reasoned about tasks while highlighting solution methods that fit with her agenda. Thus, the focus in discussions was on the numerical meaning that students' records of their measuring activity had for them. In addition, students seemed to reconceptualize their understanding of what it means to know and do mathematics as they compared and contrasted solutions. The crucial norm that became established was that of explaining and justifying solutions in quantitative terms. We find this significant because the students were not only able to reconceptualize their notions of school math in this setting, but, using Skemp's (1976) distinction, many of the students shifted from instrumental toward more relational views of doing mathematics.

## References

Cobb, P., & Yackel, E. (in press). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*.

Greeno, J. G. (1991). Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics Education, 22*, 170-218.

McClain, K. & Cobb, P. (in press). The role of imagery and discourse in supporting students' mathematical development. In M. Lampert and M. Blunk (Eds.) *Mathematical talk and school learning: What, why, and how*. Cambridge University Press.

Skemp, R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 1-7.

Thompson, A. G., & Thompson, P. W. (1996). Talking about rates conceptually, part II: Mathematical knowledge for teaching. *Journal for Research in Mathematics Education, 27*, 2-24.

# COORDINATING SOCIAL AND PSYCHOLOGICAL PERSPECTIVES TO ANALYZE STUDENTS' CONCEPTIONS OF MEASUREMENT

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This paper provides an argument for the coordination of social and psychological perspectives when analyzing students' conceptions of measurement by presenting classroom episodes taken from a four-month classroom teaching experiment. The focus of the experiment was on the development of an instructional sequence intended to support students' construction of personally-meaningful ways to reason about measurement. Against the background of the measurement studies of Piaget and the research efforts that followed, this paper presents a study in which informal individual psychological analyses were coordinated with social analyses of classroom interactions in order to provide insights into students' ways of reasoning that would subsequently inform instructional decisions.

## Introduction

Measuring plays a prominent role in both in-school and out-of-school practices. As a result, instruction in measurement is written into most elementary curricula. Accordingly, the National Council of Teachers of Mathematics *Curriculum and Evaluation Standards for School Mathematics* (1989) emphasize the importance of establishing a firm foundation in the basic underlying concepts and skills of measurement. The Standards stress that children need to "understand the attribute to be measured as well as what it means to measure" (p. 51). A large body of literature on children's conceptions of measurement has amassed over the past three decades with undoubtedly the most influential work being that of Jean Piaget and his colleagues (Piaget, Inhelder, & Szeminska, 1960). Piaget et al. identified developmental stages that they claimed children pass through as they learn to measure. As a result of Piaget's analyses, many researchers have tried to isolate the ages at which children develop certain measurement concepts. Other researchers devised training programs in order to increase the acquisition rate of measurement concepts. However, few studies have been conducted which address social and cultural influences on children's development of meaningful ways to reason about measurement. Although Piaget and his successors would not deny that social interactions are an important source of cognitive conflicts, their actual analyses focused on cognitive development without acknowledging the role that interactions with peers or interviewers played in the process.

234

In our view, learning is both an individual *and* social accomplishment with neither taking primacy over the other. Therefore, for us, investigations of students' understandings of measurement should be framed in terms of individuals' participation in the social practices of the classroom community or of the interview situations. The relationship between social and psychological processes has received increased attention in recent years but is defined differently in contrasting theoretical perspectives. Socioculturalists, for instance, give primacy to social and cultural influences. They would argue that there is a relatively *direct* link between social interactions and psychological processes (Cobb & Yackel, in press). In other words, the quality of students' thinking is generated by or derived from the social processes in which they participate. In contrast, emergent theorists characterize the link between collective and individual processes as *indirect* in that participation enables and constrains individual learning, but does not determine it. In the emergent perspective, participation in the collective mathematical practices is said to constitute the immediate social situation of learning and to provide opportunities for the possibility of learning (Cobb & Yackel, in press). Learning is both an individual and social process with neither taking primacy over the other. The social and psychological are then said to be reflexively related.

Despite the differences between the emergent and sociocultural perspectives, proponents of both theoretical perspectives claim that students' development cannot be adequately explained in cognitive terms alone; social and cultural processes must be acknowledged when explaining psychological development. It is important to note that sociocultural and emergent theories do not discount psychological analyses conducted in interviews. However, theorists from both sociocultural and emergent perspectives would argue that traditional psychological analyses characterize students' conceptual understanding independently of situation and purpose (Cobb and Yackel, in press). They would instead argue that interviews are social events where the interviewer/child system is the unit of analysis.

Interviews are not the sole means of assessing students' conceptions. Observing students as they participate in the social practices of the mathematics classroom offers an additional perspective into students' understanding. These classroom observations, especially in the context of a classroom teaching experiment, involve accounting for individuals' mathematical development as they participate in the social and cultural practices of the classroom community (Cobb, in press; Yackel, 1995). Individual students' development is analyzed in terms of their participation in and contribution to the emerging, communal mathematical practices. Further, students are seen to contribute to the evolution of the mathematical practices as they reorganize their activity while participating in these practices. As a result, the classroom teaching experiment is one site for investigating students' understandings of measurement that takes into account students' participation in social and cultural practices.

In the following sections of this paper, we will ground our discussion in a classroom teaching experiment we conducted in one first-grade classroom. We begin by first describing the intent of the instructional sequence that was utilized. Next, we describe the setting. We then present episodes from the classroom intended to highlight the importance of coordinating social and psychological analyses in supporting students' mathematical development.

## Instructional Sequence

The primary focus of the teaching experiment was to develop two closely related instructional sequences. The first sequence involved supporting students' increasingly sophisticated understanding of measuring and the second built on the measurement sequence to support students' construction of mental computation and estimation strategies for reasoning with numbers up to 100. In the case of the measurement sequence, our hope was that students would come to reason mathematically about measuring rather than become proficient with measuring procedures. This in turn made it possible for the students' activity of measuring to serve as the starting point for the second instructional sequence that focused on mental computation and estimation with two-digit numbers.

The instructional intent of the measurement sequence was formulated such that measuring was an activity in which students are physically or mentally acting on space. We wanted the activity of measuring to signify the partitioning of space into units that could be used to find a measure. Thus, measuring would not merely be a matter of iterating a unit and verbalizing the number obtained when the unit was iterated for the last time. We would want the number that results from the last iteration to signify not simply the last iteration itself but rather the result of the accumulation of the distances iterated (cf. Thompson & Thompson, 1996). For example, if students were measuring by pacing heel to toe, it was hoped that the number words they said as they paced would each come to signify the measure of the distance paced thus far rather than the single pace that they made as they said a number word. It was also hoped that the students would come to act in an environment of quantities that could be structured in different ways. For example, we wanted students to be able to interpret their measuring activity as not only a space measuring 25 feet but also five distances of five feet or two distances of ten feet and a distance of five feet, as the need arose.

## Setting

The data reported in this study were taken from a four-month classroom teaching experiment conducted in a first-grade classroom in a private school in the Nashville Metropolitan area. The 16 students, seven girls and nine boys, were primarily from upper middle-class backgrounds. The teacher was in her fifth year of teaching and was attempting to reform her practice in light of current recommendations. She viewed the members of the research team as peers with whom she could collaborate as she reflected on her changing practice.

At the beginning of the teaching experiment, interviews were conducted to assess students' mathematical understanding so that we could ensure that the instructional sequence built on their current ways of knowing. Comparisons with post-interviews indicate that, for most students, space had come to signify an object that could be partitioned, and measuring signified the accumulation of distance. Students had developed relatively sophisticated ways to think and reason about measurement. In addition, a preliminary comparison of pre- and post-interviews indicates that most of the students had developed effective thinking strategies for solving two-digit addition and subtraction problems over the course of the teaching experiment.

## Episodes from the Classroom

The instructional activities used in the teaching experiment were typically posed in the context of an ongoing narrative. To accomplish this, the teacher engaged the students in a story in which the characters encountered various problems that the students were asked to solve. The narratives served both to ground the students' activity in imagery and provided a point of reference as they explained their reasoning. A typical math session was structured as follows: (1) the teacher related the problems to be solved, (2) children worked individually or in pairs, and then (3) whole-class discussions followed.

The first narrative concerned a king who wanted to measure items in his kingdom by using his foot. However, the king did not know how he could accomplish this and requested the students' help. After students had made various suggestions, the class decided that the king could measure objects by placing one foot after another and walking the length of the item to be measured. Subsequent instructional activities included having pairs of students, each taking turns acting as king, pace the length of objects located in the classroom to determine their measure. As students engaged in the measuring activities, two distinct ways of measuring the length of different objects emerged. Some students placed one foot at the beginning of the object to be measured and counted "one" with the placement of their second foot. Other students placed their foot at the beginning of the same object and counted it as "one." We conjectured that these two ways of counting their paces were mathematically significant because students counting in the first manner did not appear to be filling space as they paced. This conjecture emerged as we conducted informal psychological analyses of "target" students during their measuring activities. These analyses indicated the need to highlight these two ways of counting as students paced. In that way, filling space might become a topic of conversation in whole class during which students could reflect on their own prior activity of measuring. It is important to note that the intent of these whole-class discussions was not to ensure that students counted their paces "the right way." Rather, we felt that if the students participated in a discussion about these two different ways of counting their paces, they would have an opportunity to reorganize their understanding about what it means to measure.

While students were discussing the two ways of counting their paces, the teacher marked the students' footsteps with tape alongside the rug that was being measured so that the class would have a record of the paces. As a consequence, many students pointed to the space between the first and second piece of tape and argued that the students reasoning in the first manner were not counting that step. Thus, a portion of the rug was not being measured. It was in the context of these types of discussions that students began to reconceptualize their prior activity.

During the whole-class discussions we attempted to make sense out of the students' discussion by bringing the social perspective to the foreground of our analysis. However, it is important to point out that our initial conjectures were informed by our prior individual psychological analyses. Thus, when making conjectures about individual students' activity, the psychological perspective came to the foreground while the social perspective faded into the background. In this way, we were coordinating a psychological and social perspective. These analyses then served to inform instructional decisions on a daily basis.

Two days later a second issue arose concerning how to account for an amount of space remaining at the end of the rug that did not make a whole foot. Often students' feet did not measure the length of the rug in a complete number of feet. During observations of their activity, we noticed that many students accounted for the extra space by turning their foot sideways in order to stay within the space bounded by the rug. For these individual students, we conjectured that pacing was an activity of filling or defining the space to be measured. In other words, they did not extend the measurement unit past the space to be measured. Other students, rather than turning their last foot sideways, simply counted the whole foot (e.g., 18 instead of 17 1/2). We, again, speculated that this was a mathematically significant issue that needed to be investigated in the context of whole class. As a result, the teacher highlighted the differing methods in the subsequent whole-class discussion.

Again, the purpose of the whole-class discussion was not to ensure that all students reasoned correctly, or even reasoned the same way. It was intended to present the students with a mathematically significant issue which they could then reason about in the context of a discussion while reflecting on their own prior activity. In this way, students were able to explain and justify the various ways they conceptualized and re-conceptualized the task. Our analyses, both social and psychological, of their current activity then served to inform our decisions about appropriate subsequent instructional activities.

## Conclusion

In the previous accounts from the classroom, we have highlighted the importance of students' participation in whole-class discussions where the problematic nature of their mathematical activity is highlighted. In doing so we have also coordinated the informal psychological analyses of students' activities and the social analysis of the subsequent whole-class discussions. It is important to note that the informal analyses were informed by both a proposed learning trajectory (Simon, 1995) and the intent of the instructional sequence. Hence, decisions about what to highlight in whole-class discussions were made in terms of the *big picture*, while the route itself took various shapes depending on how the trajectory was realized *in interaction*. This process was informed by a continuing cycle of informal individual and whole-class analyses made against the conjectured learning trajectory. This process supported students' construction of personally-meaningful ways to reason about measurement that they developed as they contributed to the emergence of the classroom mathematical practices.

## References

Cobb, P. (in press). Conducting classroom teaching experiments in collaboration with teachers. In R. Lesh and E. Kelly (Eds.), *New methodologies in mathematics and science education*. Dordrecht, The Netherlands: Kluwer.

Cobb, P. and Yackel, E. (in press). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*.

National Council of Teachers of Mathematics (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Piaget, J., Inhelder, B., & Szeminska, A. (1960). *The child's conception of geometry*. New York: Basic Books, Inc.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*, 114-145.

Thompson, A., & Thompson, P. (1996). Talking about rates conceptually, part II: Mathematical knowledge for teaching. *Journal for Research in Mathematics Education, 27*, 2-24.

Yackel, E. (1995). The classroom teaching experiment. Unpublished manuscript, Purdue University Calumet, Department of Mathematical Sciences.

# LEARNING AS SENSE-MAKING AND PROPERTY-NOTICING

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This paper outlines a framework for studying student learning and understanding. Building on a sociocultural perspective of learning that incorporates an alternative to the theory of reification, learning is described as a mutually dependent process involving personal sense making and the public negotiation of meaning, mediated by the acts of reflection and communication. The development of taken-as-shared meaning is, in part, a process of building knowledge structures through the establishment of understandings of properties of various conceptual entities. The latter understandings are a result of personal reflection and the negotiation of meaning through the act of communication. Hence, sense making and meaning making occur collectively and dependently. Results from a study discuss the kinds of algebraic thought present in the different solution strategies of 7th and 8th grade students. Analysis centered on the development of functional reasoning and the ability to abstract computation to an algebraic mode. Specifically, the analysis focused on the kinds of mathematical objects and ideas that arose during student-student talk, with particular attention to the personal and taken-as-shared nature of the properties that helped to define these objects. These data are used to provide empirical support for the framework.

## Perspectives

What is the difference between sense and meaning? Drawing on the Soviet philosophies of Vygotsky and Leont'ev, Wertsch (1991) distinguishes between sense and meaning by focusing on the personal and public aspects of activity. Lave, Murtaugh, and de la Rocha (1984) concur, stating that "sense designates personal intent, as opposed to meaning, which is public, explicit, and literal" (p. 73). Hiebert (1992) maintains that the personal act of reflection and the public act of communication are the two most important cognitive forces shaping current views of mathematical learning. These forces relate in a manner that allows one's personal reflections to mediate and be mediated by one's interactions with the environment (Bauersfeld, 1992). Hence, sense is based on one's individual reflections, whereas meaning has both a personal and public dimension.

Because the focus of this paper is on learning and knowledge construction, perhaps a more germane question would involve the differences between sense making and meaning making. A radical constructivist perspective rejects the notion that two individuals can know that they have the same knowledge, so sense making and meaning making are one in the same. von Glaserfeld (1996) states:

> "I would never say that they are wrong, but I would ask that they present a plausible model of how such sharing of meaning, and the collective generation of knowledge in language can take place." (p. 137)

Others, however, draw more separation between the two. For example, Cobb et al. (1992) describe a sociocultural perspective of learning in which students make use of external (or instructional) representations, mediated by developmental interventions of the instructor, in order to construct understandings that are either principally generated by one's sense making activities or that approximate and expand on the taken-as-shared meanings of a society. From this perspective, sense making is a mechanism by which meaning making can occur.

Kieren et al. (1995) have expanded this perspective to research settings. Their description of an *enactive* learning environment attempts to balance the aspects of the learning situation with the cognitive and social backgrounds of the participants engaged in mathematical activity. This occurs in a group interview format where the mathematical activity and research focus are mediated by researchers, participants, and setting. Kieren et al. prefer to give balance to the role of one's sense making activity and one's ability to negotiate meaning. As they state, instead of *situated* cognition or situated *cognition*, research should focus on *situated cognition*..

The theory of reification (c.f. Sfard and Linchevski, 1994 ) provides one explanation for the development of meaning. It involves the transitioning of understandings associated with actions and processes to more permanent understandings in line with mathematical objects. Slavit (in press) has articulated an alternative to this theory involving the development of meaning through an awareness of the properties of publicly negotiated and taken-as-shared objects. The ability to make sense of these properties can lead to the development of richer forms of personal meaning, such as occurs in the cognitive act of formalization. These personal meanings can then be weighed against one's perceptions of the taken-as-shared meanings of society as a whole. This perspective focuses on the development of mental constructs associated with established mathematical objects and ideas by focusing on the properties that help define them.

Algebra, as a mathematical area, has specific properties that help define it. Although algebra is a multi-faceted mathematical area, it can be viewed as an abstract form of arithmetic. For example, $a^2 - b^2 = (a+b)(a-b)$ expresses the consistent relationship between the difference of the squares of any two numbers and the product of their sum and difference. Properties associated with covariance (slope, extrema, etc.) also give meaning to the structure of algebra, particularly in regard to functional algebra.

## Method

This paper discusses one problem solving episode from a larger study that involved fourteen 7th graders and fourteen 8th graders from two middle school classrooms. The study looked at the sense making and meaning making activities of pairs of students engaged in problem solving. The students worked in pairs on two tasks (see below) for approximately 30 minutes, although some interviews went close to an hour. The students were given one copy of the first task and were told to "work the problem anyway you wish, but you may wish to work together." When it seemed that the students were at the end of their solution attempt, I questioned them on the manner in which they approached the task, and asked them to think about other ways to solve the problem. This "looking back" stage was intended to promote critical reflection by the students on their solution strategy. These procedures were repeated for the second task.

Analysis centered on the kinds of understandings that the students brought into the problem solving situation, and the kinds of understandings that the pairs utilized and constructed in their solution. Hence, the social, external, and mathematical constraints inherent in problem solving were present, but the analysis centered on the construction of understandings. Particular attention was given to the kinds of properties that the students attached to the mathematical ideas and objects which helped support their investigation. Only pencil and paper were provided:

Task 1: *Two carnivals are coming to Pullman. You and your friend decide to go to different carnivals. The carnival that you attend charges $10 to get in and an additional $2 for each ride. The carnival your friend attends charges $6 to get in, but each additional ride costs $3. If the two of you spent the same amount of money, how many rides could each of you have ridden?*

Task 2: *What digit is in the one's place of the number* $2^{[illegible]}$ $3^{4}$ $5^{[illegible]}$*?*

## Results

This paper will discuss the problem solving strategies of Tim and Molly, two eighth graders, as they worked Task 1. Students can find a solution to this task by simply adding the cost of a ride or rides to each admission until these totals are the same. However, students could also extend this strategy and find many other combinations where the amount of money spent is the same. Since there are infinitely many solutions, students could also express their answer in a manner that describes a variable relationship between the amount of rides denoting this general relationship. One way this more algebraic approach could be performed is to set the expressions $10+2a$ and $6+3b$ equal to each other, or draw the graph of this linear relationship.

The transcription of the solution strategy of Task 1 developed by Tim and Molly illustrates how the two initially constructed an understanding of the problem based on the properties of the two functions that represent the amounts of money spent by the two carnival attenders. These properties involved the initial amount (admission) and rate of increase (cost per ride). Analysis centered on their individual sense making and collective meaning making activities as they engaged in a solution attempt. This interview was chosen because of the degree of collaboration and collective sense making that occurred. Although shared meaning developed between Tim and Molly, such activity was not found in all of the interviews.

A microanalysis of specific portions of the interview reveal particular instances of sense making and property noticing that led to their solution. Consider the following segment at the onset of their solution attempt of Task 1:

Tim & Molly: (read problem, mumbling)

Molly: I don't know.

Tim: Wow (exasperated). (pause) Do you have a calculator?

DS: No no no.

Tim: OK, so right now we know that each person paid at least 10 and 6 dollars.

Molly: And then they could also go on 2 times, but it depends how many rides he'd want to go to.

Tim: How many rides can each

Molly: Actually, it depends on how many rides he'd want to go on.

Tim: No actually, um, actually it's pretty much asking what, like, sort of asking, it's almost like asking lowest common multiple, almost, or something like that, but anyway.

Molly: You actually learned something (gibingly).

Tim: That wasn't funny (good-humoredly).

The beginning portion of the interview illustrates several important aspects present throughout the interview. First, the two students felt comfortable with one another and did not appear to be nervous or affected by the camera or interview setting. Second, the students were also individually engaged in the task and actively sought an understanding of the context and solution strategy. The next few lines of the interview illustrate that the two were beginning to make use of each other's sense making activities.

Molly: One of my family's jokes. (pause) This actually depends on how many rides you went on, to go on, if you wanted to go on like 2 rides, you could spend

Tim: It depends, no, OK

Molly: for each

Tim: OK, if I wanted, if I was here and you there, right, if I wanted to go on 2 rides that would be a total of 14 dollars, for me, if you wanted to go on 2 rides then it would only be 12 dollars for you, so it would end up costing

Molly: Oh, I was mixed up, OK (laughs) well, one of them, one person didn't have to ride at all to get 10 dollars, no

Tim: OK, so what we are trying to figure out is, they spent the same amount

Molly: Yeah I know, OK, and 6 dollars, what, what adds up to being, lets see, 10, 20, (long pause), OK

Molly began to explore the problem by advancing on her initial sense of the situation, which involved an understanding of the need to consider "how many rides." She made use of the cost of admission and ride price, situational properties which correspond to the linear functional properties of y-intercept and slope. We also see that the two were beginning to construct and make use of shared meanings of the situation. Tim utilized Molly's remarks regarding the need to consider the case of going on 2 rides to begin his numerical analysis. After these computations, Molly then recognized that this would not be a desired solution and said "I was mixed up." They then returned to their individual sense making activities in the last two comments. Tim then made the following key insight:

Tim: I could go on 1 and you could go on 2 and then we'd each spend 12 dollars (very confident) would go. Yeah, (puts pencil down) I get it, there's that one.

Using the covariance properties previously explored, Tim decided to vary the number of rides for each person, changing the (2,2) case to (1,2). This produced a solution that Molly was able to immediately verbalize. Tim's personal solution became a shared and meaningful one, although Tim was clearly the initiator of nearly all of the public meaning. But this immediately changed:

DS: You spent 12. OK, so the one was 2 and the other was 1 ride?

Molly: Well you could spend more time and

Tim: You could spend alot more money, and then

Molly: Yeah

Tim: You could

Molly: One person could go on 2 rides and the other person could go on 4 rides and you'd still get the same thing.

Molly's current sense of the solution allowed her to expand the situation by linearly increasing their solutions. But as we will see, while Molly was verbalizing the meanings she constructed that led to this conjecture, Tim makes sense of Molly's remarks and challenges the meaning put forth by Molly after conducting a few computations:

DS: OK, what would they spend in that case?

Molly: Well, if you had

Tim: (writes) 6 times 30

DS: Or what makes you say that?

Tim: and if the other person goes on 4, or wait a minute, this person goes on 2.

Molly: It would be the same amount just as the first time.

DS: By the first time you mean when the one person rides 1 and the other person rides 2?

Molly: OK, one person

Tim: No it wouldn't (confidently). OK, wait so you're saying, so you're saying

Molly: OK, you go

Tim: one person goes 4 times, goes on 4 rides

Tim and Molly: and the other person goes on 2

Tim: that's 12, 14, this one's 9, 12, 15, 18, so, no that's not exactly true.

Molly: What (contentiously)?

Tim: You said one person goes on 2 and another person goes on 4.

Molly: So, but wait.

Tim: 'cause this person has to pay 6 just to get in and three for each ride

Molly: yeah

Tim: that's 4 rides, that's a total of 12 right there, plus another 6 is 18, so that's not necessarily true.

Molly: Well, um, if you take, it's 10 and 6, and then the first time, one person goes on 1 once

Tim: that's 2 dollars right there

Molly: yes, that's 2 dollars, and then another person goes on another time, that's 2 times, and then you go on it again, 6, and another 2, I guess that doesn't work. It's worked in the past for me.

Eventually their meaning making exchanges collectively led to the (4,4) solution, but no further generalizations were made by either participant.

## Conclusion

In many situations, learning is a collective process of internally constructed personal sense and externally constructed meaning. The ability to utilize one's sense to articulate meaning as well as make sense of other's stated meanings enriches the taken-as-shared network being constructed. Learning is a dynamic interplay between one's sense, one's stated meanings, and the sense one makes out of other's stated meanings. The above episode provides instances where these three activities combined to form a collective, rich understanding of the situation as well as an extensive analysis of the possible solutions. This analysis was based on personal and shared understandings of the properties that shaped the situation and the objects that helped comprise it. The taken-as-shared meanings developed were certainly a product of the two individual's sense making activities, but the development of these meanings also effected future sense making activities. These students were both willing and able to participate in this personal and shared process, and their collective sense and meaning making activities were at the heart of their learning experiences.

## References

Bauersfeld, H. (1992). Classroom cultures from a social constructivist's perspective. *Educational Studies in Mathematics, 23*, 467-481.

Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research In Mathematics Education, 23(1)*, 3-31.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A"proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education, 25(2)*, 116-140.

Hiebert, J. (1992). Reflection and communication: Cognitive considerations in school mathematics reform. *International Journal of Educational Research, 17(5)*, 439-456.

Kieren, T. E., Calvert, L. G., Reid, D. A., & Simmt, E. (1995). *Coemergence: Four enactive portraits of mathematical activity.* Paper presented at the Annual Meeting of the American Educational Research Association, San Francisco.

Lave, J., Murtaugh, M., & de la Rocha, O. (1984). The dialectical construction of arithmetic in grocery shopping. In B. Rogoff & J. Lave (Eds.), *Everyday cognition: Its development in social context* (pp. 67-94). Cambridge, MA: Harvard University.

Sfard, A., & Linchevski, L. (1994). The gains and pitfalls of reification: The case of algebra. *Educational Studies in Mathematics, 26*, 191-228.

Slavit, D. (in press). An alternate route to the reification of function. *Educational Studies in Mathematics.*

von Glaserfeld, E. (1995). *Radical constructivism.* London: Falmer Press.

Wertsch, J. V. (1981). *Voices of the mind: A sociocultural approach to mediated action.* Cambridge, MA: Harvard University Press.

# PROPORTIONAL REASONING OF EARLY ADOLESCENTS: VALIDATION OF KARPLUS, PULOS, AND STAGE'S MODEL

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The purpose of this study was to validate and extend Karplus, Pulos and Stage's (1983) model of proportional reasoning in early adolescents. Karplus et al.'s model posits four hierarchical "cognitive elements" related to numeric relationships within proportion problems. These form a scale such that students capable of solving a problem representative of a "difficult" element are also capable of solving problems representative of "easier" elements. However, Karplus et al. tested their model with only one type of problem, a "Lemonade Puzzle"—similar to Noelting and Gagne's (1980) orange juice task—which required that students make judgments about the taste of a sugar/lemon mixture. We wished to see if Karplus et al.'s scale of cognitive elements held for proportion problems drawn from other rational number subconstructs. Our data suggests that for *quotient*, *operator* and *measure* problems, the scale holds.

## Overview

In their work on proportional reasoning in adolescents, Karplus, Pulos and Stage (1983) found that sixth- and eighth-graders' performance on proportion problems is dependent upon the numerical relationships represented in the problems—that is, whether the ratios involve integers or non-integers, and whether the ratios themselves are equal or not. Consider the following four capacities: (1) comparing equal integer ratios (e.g., 2/4=6/12); (2) comparing unequal integer-noninteger ratios (e.g., 2/4<>3/7); (3) comparing equal noninteger ratios (e.g., 3/7 = 6/14); and (4) comparing non-equal noninteger ratios (e.g., 3/7<>5/8). Karplus et al. found that 80 of 83 students capable of solving level (4) problems could also solve problems at levels (3), (2) and (1). Karplus et al.'s four-level Guttman scale (Torgerson, 1958) of "cognitive elements" for proportional reasoning may interest researchers, teachers, and curricular designers, since it provides a tool for thinking about and discussing students' proportional reasoning that is perhaps analogous to Carpenter and Moser's (1982) models of how children's solution strategies interact with problem types in the additive field.

Karplus et al. tested lemonade mixture problems—a variant of Noelting and Gagne's (1980) orange juice problem. The Lemonade Puzzle poses two recipes, each of which specifies an amount of sugar and lemon are mixed to make lemonade. Students are asked to decide which recipe tastes sweeter and, if necessary, decide how much sugar or lemon would be needed to make the recipes taste the same. For example, a Lemonade Puzzle is, "John makes concentrate by using 4 spoonfuls of sugar and 10 spoonfuls of lemon juice. Mary makes concentrate by using 6 spoonfuls of sugar and 15 spoonfuls of lemon juice. Will the concentrates taste the same?" Students in the Karplus et al. study responded to a total of eight Lemonade Puzzles, each representing a different combination of integer/non-integer relations, equal or unequal ratios, and whether or not the integer relation (if present) existed within ratios or between ratios.

Our goal was to test and extend these results by varying the type of proportion problems given to children, in an effort to evaluate whether Karplus et al.'s scale of cognitive elements generalizes beyond the Lemonade Puzzle. We also wished to examine whether students' solution strategies varied according to problem type, such as their use of factoring (between-ratio) or unitizing (within-ratio) approaches.

## Rational Number Subconstructs

Proportion problems compare two rational numbers of the form *x/y*. The numbers themselves may refer to one of a number of semantic groups, termed subconstructs in the literature. Researchers posit anywhere from three to seven distinct subconstructs (c.f. Behr, Lesh, Post and Silver, 1983, 1992; Freudenthal, 1983; Kieren, 1976, 1993; Ohlsson, 1987; Rappaport, 1966; Vergnaud, 1983). For instance, the Lemonade Puzzle might be considered a mixture or ratio problem. In addition to the ratio subconstruct originally tested by Karplus et al., we tested *quotient*, *operator*, and *measure* items as well.

***Quotient.*** For Behr et al. (1983), a ratio problem fits the quotient subconstruct if the numerator/denominator pair is interpreted as an indicated division. We used partitioning problems to represent this subconstruct; these are problems where the numerator of the rational number is a quantity and the denominator is a parameter. The goal of a partitioning problem is to separate the quantity into a number of equal-sized parts (Ohlsson, 1988). The idea behind this is intuitive to children; it grows out of the idea of "fair sharing" according to Hunting (1983). Our quotient puzzle was, "Children are seated at two tables eating cookies. The children at John's table are sharing 4 cookies among 6 children. The children at Mary's table are sharing 10 cookies among 15 children. At which table does each child get more cookies, or do they get the same amount?"

***Operator.*** The operator subconstruct is a rational number interpreted as size a transformation (Kieren, 1995). The pair of numbers in a ratio can be interpreted as either duplicator and partition-reducer, multiplier and divisor, or stretcher and shrinker. For our operator puzzle we used the following: "Mary and John are taking a photography class, in which they can enlarge and reduce the photographs they take. They both start with photos of a tree that is the same size. Mary makes her tree 4 times taller, and then makes the resulting tree 6 times shorter. John makes his tree 10 times taller, and then makes the resulting tree 15 times shorter. Whose tree is now taller, Mary's or John's, or are they the same size?"

*Measure.* Measure uses a fixed reference quantity, called the unit, and a fixed partitioning parameter. Rational number as measure, then, is the numbering of these equal partitions (1/n) of the unit (Ohlsson, 1988). The focus is on unit and subdivision rather than the relationship to a whole. Our measure puzzle read as follows: "On the track at John's school, John has to run 6 laps in order to run a mile. At Mary's school, she has to run 15 laps around the track to run a mile. One day, John ran 4 laps at the track at his school. Mary ran 10 laps around the track at her school. Who ran farther, or did they run the same distance?"

*Ratio.* We used Karplus et al.'s Lemonade Puzzle (see above) to represent the ratio subconstruct.

## Methods and Procedures

*Participants.* The participants were 16 eighth-grade and 17 sixth-grade students from Midwestern middle schools. Half the students from each grade were randomly assigned to each of the four problem-task test conditions.

*Procedure.* Each student participated in an interview in which she or he was asked to solve two practice and eight test items from a single subconstruct. Quantities embedded in the problems were identical to those used by Karplus et al. (1983). Four investigators conducted the interviews, one investigator for each of the test conditions. Each interviewer coded her or his own interviews.

Interviews were conducted in accordance with a written protocol, which was reviewed for fidelity to Karplus et al.'s original protocol (S. Pulos, personal communication, May 3, 1996). All items administered to each subject conformed to a single puzzle or task type, corresponding to the participant's assigned test condition. Items in each of the test conditions used identical numerical values. Interviewers presented items by means of an illustrated while card, reading the problem text aloud. Appropriate follow-up questions were asked as necessary to probe children's strategies on each item. We tape recorded participants' responses, and collected all written work.

In pilot studies, Karplus et al. found that when participants were presented with a difficult problem at the onset of the interview, they frequently became frustrated and used irrational strategies to solve them. Subsequently, participants continued to apply irrational strategies on the remainder of the items (S. Pulos, personal communication, June 21, 1996). Consequently, Karplus et al. did not present test items in a randomized order, but instead presented them to all participants in their hypothesized order of difficulty. We presented items in this same order. Participants' explanations to the comparison questions were classified into four categories, as identified by Karplus et al., (1983): (1) incomplete or illogical, (2) qualitative, (3) additive, and (4) proportional. In addition, participants' additive and proportional response strategies were categorized, when possible, as either Within (comparing component 1 to component 2 separately for John and Mary) or Between (comparing corresponding components between John and Mary) comparisons, as identified by Noelting and Gagne (1980) and used by Karplus et al. (1983).

## Results and Discussion

The distribution of strategies used by problem type are given in Table 1 below. As Karplus et al. (1983) did, we classify strategies first as illogical (I), qualitative (Q), additive (A), or proportional (P). Additive and proportional strategies are further categorized according to whether students used a between or factoring approach (b), a within or unitizing approach, or (u) an unclassifiable approach.

Overall, students used more within (unitizing) strategies than between (factoring) strategies ($x^2=97.8$, df=1, $p<0.0001$), even on puzzles with integral between ratios ($x^2=34.0$, df=1, $p<0.0001$). This may be an artifact of the order of presentation, since all students saw puzzles with integral within ratios before integral between ratios. Only on the lemonade and photography puzzles did students tend to use between or factoring approaches.

**Table 1** *Strategy use by subconstruct*

| Strategy [a] | Quotient (cookies) | Ratio (lemonade) | Operator (photography) | Measure (running) |
|---|---|---|---|---|
| **I** | **1** | **15** | **3** | **15** |
| **Q** | **0** | **0** | **3** | **3** |
| (A-w) | 11 | 7 | 33 | 6 |
| (A-b) | 0 | 3 | 0 | 0 |
| (A-u) | 0 | 0 | 0 | 0 |
| **A** | **11** | **10** | **33** | **6** |
| (P-w) | 47 | 28 | 12 | 40 |
| (P-b) | 4 | 18 | 12 | 0 |
| (P-u) | 0 | 1 | 1 | 0 |
| **P** | **51** | **47** | **25** | **40** |
| n[b] | 8[c] | 9 | 8 | 8 |

[a] I-Illogical; Q-qualitative; A-additive; P-proportional

[b] Number of subjects. Each answered 8 items.

[c] n=7 for problem NX.

**Table 2** *Guttman scale for cognitive elements*[a]

| Element | Students[a] | Errors |
|---|---|---|
| None (no proportional reasoning) | 7 | 0 |
| Integer-integer, equal | 26 | 0 |
| Integer-noninteger, unequal | 23 | 0 |
| Noninteger-noninteger, equal | 17 | 1 |
| Integer-integer, unequal | 15 | 1 |

[a] n=33

Data from Table 1 above suggests that participants' use of proportional reasoning differed significantly among the problem types ($x^2=24.2$, df=3, $p<0.005$). Students used proportional reasoning on about 80% of the *quotient* items. For *ratio* items, about 75% of items were answered proportionally. On the *operator* items, just 40% of strategies were proportional, with participants using additive strategies for about 50% of the operator items. For the *measure* items, over 60% of the strategies were proportional. The *ratio* and *measure* problem types elicited significantly more illogical strategies—more than 20% of strategies used—than the other two contexts ($x^2=21.3$, df=3, $p<0.005$).

Combining data from all four subconstructs, the four cognitive elements formed a Guttman scale (Torgerson, 1958) in the same order as the one found by Karplus et al. (1983), and with a coefficient of scalability in excess of 0.90 (Table 2 below). Seven participants did not use proportional reasoning at all; of the remaining 26 participants, 24 fit the scale, with a coefficient of scalability in excess of 0.90. The two not fitting the scale were from measure and operator groups.

Since the scale held for the combined data, it therefore held for each of the 24 students who fit the scale. Because we administered items to students from one of the four subconstructs, the data support the hypothesis that the scale holds in each subconstruct individually.

While the sample sizes are too small to draw clear inferences within each subconstruct, we saw some interesting and suggestive patterns. All eight participants in the quotient group fit the scale, and all showed evidence of the first two cognitive elements (integer-integer equal comparisons, such as 2/4=6/12, and integer-noninteger unequal, such as 2/4<>3/7). In the ratio group, one subject showed no evidence of proportional reasoning, and the remaining eight all fit the scale. However, there were only four participants in the measure group and three participants in the operator group who fit the scale; in each of these two groups, there were three participants who showed no proportional reasoning at all, and one subject who used proportional reasoning but did not fit the scale.

Some of the differences observed among the test conditions might reflect a difference in the structure of the problem contexts. Karplus et al. (1983) designed their tasks to be free of the need for complex knowledge of physical principles. The concept of photographic enlargement and reducing may have been sufficiently unfamiliar to students to explain the low incidence of proportional reasoning elicited in the operator group. In contrast, the intuitive and preliminary nature of partitioning tasks (Hunting, 1983; Pothier & Sawada, 1983) likely explains the high incidence of proportional reasoning on the quotient tasks.

Some participants settled on a particular strategy that they developed while completing the first few items, and persisted with that strategy on later items. This is consistent with the findings of Karplus et al. (S. Pulos, personal communication, June 21, 1996) that participants' overall strategy use was based on the strategies they used in the earliest items. Since the test items were presented to participants in a pre-conceived order of difficulty, which corresponds to the scale derived from the data, the distribution of strategies and the scale of cognitive elements derived from them may be confounded with item order. Future work should examine the effect of changing the order of presentation of the test items (for example, presenting some between-ratio-integral items before some within-ratio-integral items). The effects of reversing the order in which the four numbers are presented in each item should also be investigated. Similarly, student performance with multiple problem types should be explored.

## References

Behr, M. J., Lesh, R., Post, T., & Silver, E. A. (1983). Rational-number concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematical concepts and processes*, (pp. 91-126) Orlando, FL: Academic Press.

Behr, M., Harel, G., Post, T., & Lesh, R. (1992). Rational number and proportion. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296- 333). New York: Macmillan.

Carpenter, T.P. and Moser, J.M.(1982). The development of addition and subtraction problem-solving skills. In T.P. Carpenter, J.M. Moser, and T.A. Romberg, (Eds.), *Addition and Subtraction: A Cognitive Perspective*. Lawrence Erlbaum: Hillsdale, NJ.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Boston: D. Reidel.

Hunting, R.P. (1983). Alan: A case study of knowledge of units and performance with fractions. *Journal for Research in Mathematics Education, 14*,182-197.

Karplus, R., Pulos, S., & Stage, E.K. (1983). Proportional reasoning of early adolescents. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes*. (pp. 45-89). Orlando, FL: Academic Press.

Kieren, T. (1976). On the mathematical, cognitive, and instructional foundations of rational numbers. In R. Lesh (Ed.), *Number and measurement: Papers from a research workshop* (pp. 101-144). Columbus, OH: ERIC/ SMEAC.

Kieren, T. (1995). Creating spaces for learning fractions. In J. Sowder & B. Schappelle (Eds.), *Providing a foundation for teaching mathematics in the middle grades* (pp. 31- 65). Albany, NY: State University of New York Press.

Noelting, G., & Gagne, L. (1980). The development of proportional reasoning in four contexts. In R. Karplus (Ed.), *Proceedings of the Fourth International Conference for the Psychology of Mathematics Education*. Berkeley, CA: Lawrence Hall of Science, 1980.

Ohlsson, S. (1988). Mathematical meaning and applicational meaning in the semantics of fractions and related concepts. In J. Hiebert & M. Behr (Eds.), *Number concepts and operation in the middle grades*. (pp. 53-92). Reston, VA:National Council of Teachers of Mathematics.

Pothier, Y. & Sawada, D. (1983). Partitioning: The emergence of rational number ideas in young children. *Journal for Research in Mathematics Education, 14*,307-317.

Rappaport, D. (1966). *Understanding and teaching elementary school mathematics.* New York: John Wiley and Sons, Inc.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematical concepts and processes* (pp. 127-174). New York: Academic Press.

# A VYGOTSKIAN MODEL FOR UNFOLDING, FORMULATING, AND SOLVING MATH PROBLEMS FROM CHILDREN'S NARRATIVES OF EXPERIENCE

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This Vygotskian model describes one way in which teachers build on children's spontaneous concepts of various situations to facilitate children's construction of formal mathematical concepts, symbolism, and problems. This specifies in the area of early mathematical problem solving Vygotsky's hypothesized inter-connection of spontaneous concepts and formal scientific school concepts.

This process is one in which the teacher leads the class in a mathematizing process focused on one child's story and/or drawing that is based on personal experience. The teacher first asks other children to retell the story in their own words and to ask and answer questions about the story. This earliest phase facilitates listening, memory, and participation as well as understanding. Next, the teacher has the children ask questions about mathematical aspects of the story. From these elements, the teacher poses a mathematized story that still possesses complex real-world aspects but that omits many of the non-mathematical elements of the original story. Several children retell this mathematized story and ask and answer questions about it so that children understand this new version. Children pose various questions about a particular situation that occurs within the context of the story; these questions form different kinds of problem situations. The final result of these questions is a math problem that resembles typical school word problems.

Students then set out to solve the problem. Teachers encourage students to solve it in different ways. While some students quickly solve the problem using fingers, equations, or mental methods, some need more concrete experiences to see or feel the answer. Even when some students have the answer automatically they are asked to draw or model the problem situation so that they can reflect on their mathematical thinking and be able to explain the process more thoroughly. This is especially true when they are exploring a new type of math word problem. The models (or drawings) serve as a scaffold to help the child remember the steps that s/he took to solve the problem. Children also act out problems, especially when the story is about selling experiences. A small group or a pair can act the situation out in front of the class.

Learning within a context, especially if that context comes directly from their lives, children are able to construct methods that are comprehensible and meaningful to them. Our goal is to empower students to apply their math skills and experience to their daily lives, as well as to encourage and support them in generating work on their own.

The research reported in this paper was supported by the National Science Foundation under Grant No. RED 935373. The opinions expressed in this paper are those of the author and do not necessarily reflect the views of NSF.

# DIFFERENT APPROACHES TO THE SEMIOTIC SYSTEMS OF REPRESENTATION: REPRESENTATION IN A MATHEMATICAL ACTIVITY

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The discussion we propose is a reflection focused on the necessity on the part of the students to coordinate different representations that are in different semiotic systems of representation in the construction of mathematical knowledge. That is, in order to differentiate a mathematical object from his representation, it is necessary for the student to represent that mathematical object with at least two different representations.

Duval (1993, p. 40) quotes: "a semiotic system could be a representation register, when it must permit three cognitive activities related to the semiosis: (1) the presence of an identifiable representation; (2) *the treatment* of a representation which is the transformation of the representation *within the same register where it has been formed*; and *(3) the conversion* of a representation which is the *transformation* of the representation in other representation of another register in which it conserves the totality or part of the meaning of the initial representation."

Analyzing the work of different authors related to semiotic systems of representation, and our research on epistemological obstacles, an aspect that it is not clear on these authors is the place of the error or the presence of an epistemological obstacle in this context. For the construction of mathematical concepts, on the one hand, it is necessary for the study of epistemological obstacles; on the other hand, it seems also necessary to develop a theory of knowledge to support a theory of representation (i.e., Semiosis and Noesis). The use in the classroom to provoke in the students the coordination, out of contradictions, different semiotic systems of representations related to the concept immerse in a problem situation.

# JILL'S USE OF DEDUCTIVE REASONING: A CASE STUDY FROM GRADE 10

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Mathematical reasoning, especially deductive reasoning, has been identified in curriculum reform documents and the research literature as an important area for additional research. This case study is part of a long term school based research project on students' use of deductive reasoning and the contexts in which such reasoning occurs. The student who is the focus of the case study is from a grade 10 classroom in which a problem solving/discovery approach was taken to the teaching of coordinate and Euclidean geometry.

The key features of the contexts in which Jill used deductive reasoning include the activities presented in the class, and the nature of her interactions with her peers, her teacher and the researchers. An important commonality of these features is that they encouraged the use of deductive reasoning to *explain*, rather than verify or explore mathematics.

The activities were designed to allow problem solving, pattern noticing and deductive reasoning to be used in learning the content specified by the curriculum. In cases where deductive reasoning was explicitly called for (such as explaining geometrical principles), more deductive reasoning took place than in contexts related to objectives unsuited to deductive reasoning (such as determining an equation from a graph).

In some cases the teacher, or researchers acting in the role of a teacher, provided verbal prompts that resulted in extended deductive arguments. Such prompts extended the activities and were interrogatory rather than instructive. There was an explicit message from the teacher that the students should explain any assertions they made, which translated into a social context which contributed to some occasions for deductive reasoning. In other cases the teacher or researcher made a comment that interrupted activity in which deductive reasoning was taking place or provided instructions which discouraged deductive reasoning.

The two main factors related to peers as contexts for deductive reasoning were the peers' interest in the mathematical activity, and the mathematical background they had to draw on. Interactions with peers who were interested and knowledgeable led to more use of deductive reasoning that interactions with peers who were disinterested and less knowledgeable.

# THE DEVELOPMENT OF CHILDREN'S UNDERSTANDING OF ADDITIVE COMMUTATIVITY

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The research before 1975 left us with little substantial knowledge of children's understanding of additive commutativity. Although mathematics educators have long recommended using this principle to help children master basic addition fact, there is, in fact, little methodologically sound evidence to support this contention.

Research from 1975 to 1989 indicated that children understood commutativity before formal instruction on the principle and as early as kindergarten. It suggested that they discover this regularity as a result of their computational experience, but that this principle was not a necessary condition for inventing computational strategies that disregarded addend order.

In the 1990s, Resnick (1992) proposed that commutativity is derived from a general understanding of additive composition, and others (e.g., Ganetsou & Cowan, 1995) concluded that this principle develops independently of computational experience but serves as a basis for strategies that disregard addend order. Unfortunately, the evidence for these propositions is either ambiguous or methodologically flawed.

Consistent with Resnick's (1992) model, commutativity may evolve from a relatively weak schema to a relatively strong one. Initially, however, addition may be linked to an order-constrained, rather than an order-indifferent, part-whole schema. Computational experience may help to connect this operation to a part-whole schema characterized by commutativity.

## References

Ganetsou, E., & Cowan, R. (1995, September). The development of commutativity. Paper presented at the Conference on Language and Mathematical Thinking, London, England.

Resnick, L. B. (1992). From protoquantities to operators: Building mathematical competence on a foundation of everyday knowledge. In G. Leinhardt, R. Putnam, & R. A. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching* (pp. 373-425). Hillsdale, NJ: Erlbaum.

# THE EFFECTS OF PROBLEM SIZE ON CHILDREN'S UNDERSTANDING OF ADDITIVE COMMUTATIVITY

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The ability to solve problems regardless of number size is taken as evidence of a general principle. Resnick (1992) argued that an understanding of additive commutativity is initially context-bound and tied to number size. Existing evidence of size effects is mixed and limited to addends less than 10.

This study examined the performance of 24 kindergartners and 24 first graders on commutativity tasks involving addends 3 to 9, 13 to 19, and 26 to 59. The order of the levels was counterbalanced. Within each level, there were two commuted trials (e.g., Big Bird has 9 cookies and he got 7 more. If Cookie Monster already has 7 cookies, how many more does he need to have the same number as Big Bird?) and one noncommuted trial to check for response biases. These trials were presented in random order.

The data were analyzed using a conservative criterion (subjects responded correctly and *differentially* to commuted and noncommuted trials) and a liberal criterion (subjects responded correctly to commuted trials). Although qualitative analyses indicated that addend size seems to affect some children, a 2 (Group: K vs. 1) x 3 (Addend size: small vs. teen vs. large) ANOVA using the conservative criterion did not detect a significant difference for grade level or addend size. The ANOVA using the liberal criterion did approach significance ($F(1.86,94) = 2.81$, $p = .07$, MsE = .09). The results suggest that, for most children, commutativity is understood as a relatively general concept. They also indicate that previous research that did not control for a response bias (e.g., Bermejo & Rodriguez, 1993) may have overestimated knowledge of commutativity.

## Reference

Bermejo, V., Rodriguez, P. (1993). Children's understanding of the commutative law of addition. *Learning and Instruction, 3*, 55-72.

Resnick, L. B. (1992). From protoquantities to operators: Building mathematical competence on a foundation of everyday knowledge. In G. Leinhardt, R. Putnam, & R. A. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching* (pp. 373-425). Hillsdale, NJ: Erlbaum.

# KINDERGARTNERS' UNDERSTANDING OF ADDITIVE COMMUTATIVITY WITHIN THE CONTEXT OF WORD PROBLEMS

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Baroody and Gannon (1984) proposed that children's understanding of additive commutativity progresses through several stages that are based on a unary understanding of addition (change meaning) before developing a "true" understanding of commutativity based on a binary conception (part-whole meaning). Resnick (1992) also proposed that an understanding of commutativity progressively becomes more sophisticated. However, she implies that children have a unary and binary conception of addition from the earliest stages of development.

In this study, 25 kindergartners' understanding of additive commutativity was investigated using performance on tasks involving two types of addition word problems. A child's unary understanding of commutativity was tested using "change-add-to" word problems.[1] A child's binary understanding was tested using "part-part-whole" word problems.[2] In both cases, children were asked if commuted and noncommuted situations would result in the same sum.

Fourteen children (56%) were successful on both additive commutativity tasks, seven (28%) were unsuccessful on both. Four (16%) were successful on the binary task only, which is inconsistent with a value of zero implied by Resnick's model ($p = .16$, 95% CI: $.04 \leq p$, one-tail). The data were also inconsistent with, but not sufficient to reject, the Baroody-Gannon model (Binomial Test, $p = .0625$). The results do suggest a weak commutative permission that is used inconsistently across tasks.

## References

Baroody, A. J., & Gannon, K. E. (1984). The development of commutativity principle and economical addition strategies. *Cognition and Instruction, 1*, 321-329.

Resnick, L. B. (1992). From protoquantities to operators: Building mathematical competence on a foundation of everyday knowledge. In G. Leinhardt, R. Putnam, & R. A. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching* (pp. 373-425). Hillsdale, NJ: Erlbaum.

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[1] This problem type implies a physical action in which a starting amount is increased by a second amount.

[2] This problem type represents a static situation in which the two amounts are present from the start and are combined to create the whole.

# CHILDREN'S REPRESENTATIONS OF MULTIPLICATION

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This study examines sixth graders' representations of multiplication. Data was collected by way of individual interviews in which the subjects were asked to complete various tasks, such as a card sort, using a manipulative, drawing pictures, writing a story problem, etc. The researchers also used a Piagetian task to study development from additive to multiplicative reasoning. The results led researchers to conclude that children have many different representations of multiplication, some of them correct and some incorrect. The most common correct representation was representing multiplication as repeated addition. The multiplication story problems written by the participants ranged from addition to multiplication to nonsensical. The researchers compared each subject's story problems with their representation of multiplication to construct a picture of each subject's overall conceptions of multiplication.

## References

Clark, F. and Kamii, C. (1996). Identification of multiplicative thinking in children in grades 1-5. *Journal for Research in Mathematics Education, 27*(1), 41-51.

Harel, G., Ed.; Confrey, Jere,(Ed.) (1994). *The Development of Multiplicative Reasoning in the Learning of Mathematics.* New York: New York University Press.

Greer, B.(1992). Multiplication and division as models of situations. In *Handbook of research on mathematics teaching and learning: A Project of the national council of teachers of mathematics.*
(pp.276-295). New York: Macmilan Publishing Co.

# Appendix

# Index of Authors

| | | |
|---|---|---|
| Abramovich | Sergei | Technology |
| Acuña | Claudia | Geometry and Geometric Thinking |
| Aguilar | Verónica | Algebra and Algebraic Thinking |
| Ajose | Sunday A. | Teacher Beliefs |
| Anderson | Dianne S. | Teacher Education |
| Arellano | Antonio | Functions and Graphs |
| Armour-Thomas | Eleanor | Teacher Knowledge |
| Artzt | Alice F. | Teacher Knowledge |
| Arvold | Bridget | Teacher Beliefs |
| Baroody | Arthur J. | Student Knowledge |
| Baroody | Arthur J. | Student Knowledge |
| Baroody | Arthur J. | Student Knowledge |
| Bastable | Virginia | DG3-Teacher Education |
| Battista | Michael T. | Technology |
| Becker | Joanne Rossi | Reform |
| Bell | Karen | Assessment |
| Benson | Carol | Algebra and Algebraic Thinking |
| Berenson | Sarah | Algebra and Algebraic Thinking |
| Berenson | Sarah | Technology |
| Berenson | Sarah B. | Teacher Knowledge |
| Berlin | Donna F. | Research Methods & Instrum. |
| Birgisson | Gudmundur | Geometry and Geometric Thinking |
| Bishop | Joyce | Algebra and Algebraic Thinking |
| Blanton | Maria | Student Knowledge |
| Blanton | Maria | Technology |
| Blanton | Maria L. | Teacher Knowledge |
| Blume | Glendon W. | Technology |
| Blume | Glendon W. | Technology |
| Bolte | Linda A. | Assessment |
| Borrow | Caroline | Technology |
| Bouniaev | Mikhail M. | Advanced Mathematical Thinking |
| Bowers | Janet | Student Knowledge |
| Bowman | Anita H. | Teacher Beliefs |
| Brahier | Daniel J. | Assessment |
| Brenner | Betsy | DG4-Language |
| Brenton | Lawrence | Advanced Mathematical Thinking |
| Bright | George W. | Teacher Beliefs |
| Burgeson | Joel | Teacher Education |
| Burghardt | Birch | Language/Discourse/Socio-Cultural |
| Bush | William | DG5-Teacher Education |
| Cai | Jinfa | Problem Solving |
| Carlson | Marilyn | DG9-Teacher Education |

|  |  |  |
|---|---|---|
| Carlson | Marilyn P. | Research Methods & Instruments |
| Carter | Jack | Curriculum Reform |
| Cenet | Jean-Marc | Geometry and Geometric Thinking |
| Chazan | Daniel | Algebra and Algebraic Thinking |
| Cifarelli | Victor | Algebra and Algebraic Thinking |
| Clement | Lisa | Functions and Graphs |
| Cobb | Paul | Student Knowledge |
| Cobbs | Georgia A. | DG2-Teacher Education |
| Cobbs | Georgia A. | Teacher Beliefs |
| Confrey | Jere | Technology |
| Connelly | Richard P. | Assessment |
| Contreras | José N. | Teacher Knowledge |
| Contreras | José N. | Data, Probability, & Statistics |
| Contreras | José N. | Teacher Knowledge |
| Contreras | José N. | Teacher Knowledge |
| Cooney | Thomas J. | Assessment |
| Cooper | Martin | Problem Solving |
| Cordero | Francisco | Advanced Mathematical Thinking |
| Cordero | Francisco | Advanced Mathematical Thinking |
| Cordero | Francisco | Advanced Mathematical Thinking |
| Cordero | Francisco | Functions and Graphs |
| Coulombe | Wendy | Algebra and Algebraic Thinking |
| Cramer | Kathleen | Teacher Education |
| Cramer | Kathleen | DG5-Teacher Education |
| Crawford | Ann R. | Instruction |
| Cuellar | Raul | Advanced Mathematical Thinking |
| Curcio | Frances R. | International Studies |
| Day | Roger | DG1-Technology |
| de la Cruz | Yolanda | Language/Discourse/Socio-Cultural |
| de la Cruz | Yolanda | DG7-Language |
| de la Cruz | Yolanda | DG4-Language |
| de Silva | Rapti | Geometry and Geometric Thinking |
| de Silva | Rapti | Instruction |
| DeMarois | Phil | Functions and Graphs |
| DiazBarriga | Eugenio | Technology |
| Ebert | Dave | Student Knowledge |
| Edwards | Barbara S. | Advanced Mathematical Thinking |
| Edwards | Laurie D. | Language/Discourse/Socio-Cultural |
| Edwards | Thomas G. | Teacher Beliefs |
| Edwards | Thomas G. | Advanced Mathematical Thinking |
| Elliott | Rebekah L. | Language/Discourse/Socio-Cultural |
| Elliott | Rebekah L. | Teacher Knowledge |
| Erickson | David R. | Teacher Beliefs |
| Erickson | David R. | DG2-Teacher Education |
| Esquinca | Miguel Solis | Advanced Mathematical Thinking |

| | | |
|---|---|---|
| Feikes | David | Teacher Knowledge |
| Fernández | Maria L. | Teacher Education |
| Ferrucci | Beverly | Curriculum Reform |
| Filho | Jose Castro | Technology |
| Flanagan | Karen | Technology |
| Flanagan | Karen | Technology |
| Fleener | M. Jayne | Teacher Beliefs |
| Fonseca | Jose D. | Language/Discourse/Socio-Cultural |
| Ford | Janet E. | Data, Probability, & Statistics |
| Fox | Tom | Advanced Mathematical Thinking |
| Friel | Susan N. | Teacher Education |
| Fuson | Karen C. | DG4-Language |
| Fuson | Karen C. | Language/Discourse/Socio-Cultural |
| Galindo | Enrique | Geometry and Geometric Thinking |
| Gellings | Linda | Student Knowledge |
| Geoghegan | Noel | Instruction |
| Glanfield | Florence | Teacher Knowledge |
| Gober | Deborah A. | Teacher Beliefs |
| Gober | Deborah A. | Teacher Beliefs |
| Goodson-Espy | Tracy | Algebra and Algebraic Thinking |
| Gravemeijer | Koeno | Student Knowledge |
| Greeno | James G. | Plenary |
| Hadfield | Oakley D. | Teacher Education |
| Haller | Susan | Teacher Education |
| Hanzsek-Brill | Melissa B. | Teacher Education |
| Harnisch | Delwyn L. | Assessment |
| Hauger | Garnet S. | Advanced Mathematical Thinking |
| Heid | M. Kathleen | Technology |
| Heid | M. Kathleen | Technology |
| Heinz | Karen | Reform |
| Hensien | Sarah | Teacher Beliefs |
| Herbst | Patricio | Problem Solving |
| Hernández | Daniel | Advanced Mathematical Thinking |
| Herrelko | Janet | Algebra and Algebraic Thinking |
| Herzig | Abbe H. | Student Knowledge |
| Hitt-Espinosa | Fernando | Student Knowledge |
| Hodgson | Ted | Advanced Mathematical Thinking |
| Hollar | Jeannie | DG8-Teacher Beliefs |
| Hollar | Jeannie | Teacher Knowledge |
| Inoue | Noriyuki | Geometry and Geometric Thinking |
| Iseri | Linda | Technology |
| Iseri | Linda | Technology |
| Ivey | Kathy M. C. | Student Knowledge |
| Jones | Graham A. | Data, Probability, & Statistics |
| Kalchman | Mindy | Functions and Graphs |

| | | |
|---|---|---|
| Kaplan | Rochelle G. | Language/Discourse/Socio-Cultural |
| Keen | Virginia L. | Teacher Education |
| Kenney | Patricia | Assessment |
| Kerr | Kenneth | Technology |
| Kerr | Kenneth | Technology |
| Kersaint | Gladis | Algebra and Algebraic Thinking |
| Khisty | Lena | DG7-Language |
| Kieren | Thomas | Student Knowledge |
| Kinzel | Margaret | Reform |
| Kitchen | Richard S. | Reform |
| Kitchens | Anita N. | Teacher Knowledge |
| Kitchens | Anita N. | DG8-Teacher Beliefs |
| Klanderman | David B. | Functions and Graphs |
| Knuth | Eric J. | Language/Discourse/Socio-Cultural |
| Knuth | Eric J. | Language/Discourse/Socio-Cultural |
| Knuth | Eric J. | Teacher Knowledge |
| Krumpe | Norm | Geometry and Geometric Thinking |
| Krussel | Libby | Teacher Education |
| Krussel | Libby | DG2-Teacher Education |
| LaBerge | Victoria | Teacher Education |
| Lambdin | Diana V. | DG6-Research Methods & Instruments |
| Langer | Ellen | Language/Discourse/Socio-Cultural |
| Langrall | Cynthia W. | Algebra and Algebraic Thinking |
| Langrall | Cynthia W. | Data, Probability, & Statistics |
| Layman | John | Functions and Graphs |
| Lesh | Richard | DG5-Teacher Education |
| Li | Yeping | Curriculum Reform |
| Licon-Khisty | Lena | DG4-Language |
| Licon-Khisty | Lena | Teacher Beliefs |
| Lillard | Eileen | Instruction |
| Lin | Pi-Jen | Teacher Education |
| Lloyd | Gwendolyn M. | Reform |
| Lo Cicero | Ana | DG7-Language |
| Lo Cicero | Ana Maria | Student Knowledge |
| Lubinski | Cheryl | Algebra and Algebraic Thinking |
| Lubinski | Cheryl A. | Instruction |
| Lutz | Mike | Geometry and Geometric Thinking |
| Maloney | Alan | Technology |
| Marge | Joan J. | Instruction |
| Marshall | James A. | Technology |
| Marshall | James A. | Technology |
| Marshall | James A. | Data, Probability, & Statistics |
| Martínez-Cruz | Armando M. | Functions and Graphs |
| Masingila | Joanna | Geometry and Geometric Thinking |
| Masingila | Joanna | DG4-Language |

| | | |
|---|---|---|
| McClain | Kay | Student Knowledge |
| McClain | Kay | Student Knowledge |
| McCrone | Sharon | Language/Discourse/Socio-Cultural |
| McGehee | Jean | DG9-Teacher Education |
| McKinley | Michelle | Teacher Education |
| McLeod | Douglas B. | Data, Probability, & Statistics |
| McNeal | Betsy | Teacher Education |
| Meel | David E. | Advanced Mathematical Thinking |
| Mellissinos | Melissa | Data, Probability, & Statistics |
| Mesa | Vilma | Problem Solvings |
| Mewborn | Denise S. | Teacher Beliefs |
| Mgembelo | Joyce | Student Knowledge |
| Mick | Harold W. | Instruction |
| Mokhtari | Kouider | Teacher Beliefs |
| Molina | David | DG9-Teacher Education |
| Mooney | Edward | Language/Discourse/Socio-Cultural |
| Moschkovich | Judit | Curriculum Reform |
| Moschkovich | Judit | DG4-Language |
| Movshovitz-Hadar | Nitsa | Advanced Mathematical Thinking |
| Norwood | Karen S. | Student Knowledge |
| Oktaç | Asuman | Advanced Mathematical Thinking |
| Oktaç | Asuman | Advanced Mathematical Thinking |
| Oktaç | Asuman | Functions and Graphs |
| Oliveros | Rodolfo | Advanced Mathematical Thinking |
| Ortega | Germán | Advanced Mathematical Thinking |
| Osorio | Francisco | Advanced Mathematical Thinking |
| Otto | Albert | Algebra and Algebraic Thinking |
| Otto | Albert D. | Instruction |
| Pack | Miriam | Teacher Beliefs |
| Parker | Michelle | Language/Discourse/Socio-Cultural |
| Pehkonen | Erkki | International Studies |
| Pence | Barbara J. | Reform |
| Peressini | Dominic D. | Language/Discourse/Socio-Cultural |
| Pinzka | Marlene | Teacher Education |
| Porzio | Donald | DG1-Technology |
| Porzio | Donald T. | Advanced Mathematical Thinking |
| Post | Thomas | Teacher Education |
| Post | Thomas R. | DG5-Teacher Education |
| Pourdavood | Roland G. | Teacher Beliefs |
| Presmeg | Norma C. | Curriculum Reform |
| Preston | Ronald V. | DG6-Research Methods & Instruments |
| Rachlin | Sid | |
| Reid | David A | DG5-Teacher Education |
| Reinke | Kathryn S. | Student Knowledge |
| Remillard | Janine T. | Teacher Beliefs |

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|---|---|---|
| Reynolds | Anne | Instruction |
| Reys | Barbara | Instruction |
| Reys | Robert | DG5-Teacher Education |
| Rich | Beverly S. | DG5-Teacher Education |
| Riggs | Laurie J. | Instruction |
| Risnes | Martin | Technology |
| Ritchhart | Ron | Research Methods & Instruments |
| Roche | Erin | Language/Discourse/Socio-Cultural |
| Rowell | Cylle | Language/Discourse/Socio-Cultural |
| Rowell | Denise | Teacher Beliefs |
| Santos-Trigo | Manuel | Student Knowledge |
| Santos-Trigo | Manuel | Advanced Mathematical Thinking |
| Schifter | Deborah | Problem Solving |
| Schultz | Karen A. | DG3-Teacher Education |
| Shama | Gilli | Teacher Beliefs |
| Shama | Gilli | Functions and Graphs |
| Shaw | Kenneth L. | Advanced Mathematical Thinking |
| Shimizu | Yoshinori | Teacher Beliefs |
| Shirley | Graeme | Geometry and Geometric Thinking |
| Shroyer | Janet | Problem Solving |
| Shunta | Norma | Teacher Education |
| Siebert | Daniel | Teacher Education |
| Simmt | Elaine | Algebra and Algebraic Thinking |
| Simon | Martin A. | Student Knowledge |
| Simon | Martin A. | Teacher Education |
| Sinicrope | Rose | Reform |
| Siongco | Rosanna | Instruction |
| Slavit | David | Instruction |
| Smith | Cynthia M. | Student Knowledge |
| Smith | Erick | Assessment |
| Smith III | John P. | Language/Discourse/Socio-Cultural |
| Snook | Kathleen G. | Curriculum Reform |
| Stallings | Lynn | Advanced Mathematical Thinking |
| Steele | Diana F. | Technology |
| Stefanova | Natalia L. | Reform |
| Stephan | Michelle | International Studies |
| Stephens | Gwendolyn | Student Knowledge |
| Stump | Sheryl L. | Technology |
| Swafford | Jane | Advanced Mathematical Thinking |
| Thomas | Christine D. | Algebra and Algebraic Thinking |
| Thornton | Carol A. | Teacher Beliefs |
| Tiilikainen | Sirpa | Data, Probability, & Statistics |
| Tiilikainen | Sirpa | Student Knowledge |
| Tiilikainen | Sirpa | Student Knowledge |
| Timmerman | Maria | Student Knowledge |

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|---|---|---|
| Törner | Günter | Teacher Knowledge |
| Tzur | Ron | International Studies |
| Tzur | Ron | Reform |
| Vacc | Nancy N. | Teacher Knowledge |
| Van Zoest | Laura R. | Teacher Beliefs |
| Vidakovic | Draga | Teacher Education |
| Walen | Sharon B. | Technology |
| Warfield | Janet | Teacher Education |
| Wetherill | Karen | Assessment |
| White | Arthur L. | Instruction |
| Wiest | Lynda R. | Research Methods & Instruments |
| Wilkins | Jesse L. M. | Problem Solving |
| Wilkins | Jesse L. M. | Student Knowledge |
| Wilkins | Jesse L. M. | Student Knowledge |
| Williams | Steven R. | Student Knowledge |
| Willner | Elizabeth | Teacher Beliefs |
| Wilson | Linda | Teacher Beliefs |
| Wilson | Melvin (Skip) | Assessment |
| Wohlhuter | Kay A. | Reform |
| Wortham | Donald W. | Teacher Beliefs |
| Zollman | Alan | Student Knowledge<br>Teacher Education |